Planar Ising magnetization field I. Uniqueness of the critical scaling limit

Federico Camia Christophe Garban Charles M. Newman

Abstract

The aim of this paper is to prove the following result. Consider the critical Ising model on the rescaled grid $a\,\mathbb{Z}^2$. Then, the renormalized magnetization field

$$\Phi^a := a^{15/8} \sum_{x \in a\,\mathbb{Z}^2} \sigma_x \delta_x \,,$$

seen as a random distribution (i.e., generalized function) on the plane has a unique scaling limit as the mesh size $a \searrow 0$. The limiting field is conformally covariant and will be shown in [CGN] to be non-Gaussian.

Contents

1 Introduction

1.1 Overview and main results

Consider the Ising model on the rescaled grid $a\,\mathbb{Z}^2$ at the *critical temperature* $\beta_c = \beta_c(\mathbb{Z}^2)$. (We refer for example to [Gri06] for a nice introduction to the Ising model.) We will be interested in the following object.

Definition 1.1. *The renormalized* **magnetization field** Φ^a, *a random distribution on the plane, is*

$$\Phi^a := \Theta_a \sum_{x \in a\,\mathbb{Z}^2} \sigma_x \, \delta_x \, ,$$

where Θ_a is a well-chosen renormalization factor. (In fact we will use a slightly modified version, see Definition 2.1).

As we will see later in Definition 1.7, we will fix [1] $\Theta_a := a^{15/8}$.

Our main theorem can be stated as follows.

Theorem 1.2 (Scaling limit)**.** *The magnetization field Φ^a converges in law as the mesh size $a \searrow 0$ to a limiting random distribution Φ^∞. The convergence in law holds in the Sobolev space $\mathcal{H}^{-3}$ under the topology given by $\|\cdot\|_{\mathcal{H}^{-3}}$. (See Section 2.)*

In fact our result holds for any simply connected [2] domain $\Omega \subset \mathbb{C}$. More precisely, consider a simply connected domain Ω in the plane which contains the origin, and let Ω_a denote its approximation by the grid $a\,\mathbb{Z}^2$ of mesh size a, i.e., $\Omega_a := \Omega \cap a\,\mathbb{Z}^2$. (The approximation might not be simply connected anymore, so in this case we keep only the connected component of the origin). Consider the Ising model in Ω_a with $+$ boundary conditions, at the **critical temperature** β_c (we will also analyze the case of free boundary conditions). The above definition of (renormalized) magnetization field easily extends to this setting:

$$\Phi^a_\Omega (= \Phi^a) := \Theta_a \sum_{x \in \Omega_a} \sigma_x \, \delta_x \, .$$

[1] This particular choice assumes Wu's result [MW73, Wu66]. Note that this choice may be debatable. For example, the authors of [CHI12] don't assume Wu's result. Without such an assumption, our results remain valid with Θ_a defined more implicitly. See Definition 1.7 and Section 6.

[2] In principle the results do not require the domain to be simply connected but this assumption allows us to directly use various results from the literature that so far were proved only in the simply connected case.

Theorem 1.3. *Let Ω be a bounded simply connected domain of the plane with a smooth boundary. Consider the critical Ising model with + or free boundary conditions in Ω_a. Then, the magnetization field $\Phi^a_\Omega = \Phi^a$ converges in law as the mesh size $a \searrow 0$ to a limiting random distribution $\Phi^\infty_\Omega = \Phi^\infty$. The convergence in law holds in the Sobolev space $\mathcal{H}^{-3} = \mathcal{H}^{-3}(\Omega)$ under the topology given by $\|\cdot\|_{\mathcal{H}^{-3}}$. (See Section 2).*

Note that the following corollary on the renormalized magnetization random variable follows easily from the above result. We state it in a similar way as one would state a central limit theorem.

Corollary 1.4. *Consider the critical Ising model in the $N \times N$ square Λ_N with + boundary conditions on $\partial\Lambda_N$. Then, the random variable*

$$\frac{1}{N^{15/8}} \sum_{x \in \Lambda_N} \sigma_x$$

converges in law as $N \to \infty$.

We will list a number of properties satisfied by the limiting fields $\Phi^\infty_{\mathbb{C}}$, Φ^∞_Ω, m^∞ (see Definition 1.5 below) in Section 7. Most of them will be proved in [CGN]. In particular, we will prove in [CGN] that these limiting fields are not Gaussian (with $\exp(-c\,x^{16})$ tail behavior).

The field $\Phi^\infty_{\mathbb{C}}$ and some of the related near-critical scaling limit fields (e.g., as in Section 7 below) are ϕ^4 Euclidean (quantum) field theories—see, e.g., [Si74].

Definition 1.5 (Renormalized magnetization)**.** *For any simply connected domain Ω with boundary condition ξ on $\partial\Omega$ (which in this paper will be either + or* free*), let m^a_Ω be the renormalized magnetization in the domain Ω defined by*

$$\begin{aligned} m^a_\Omega &:= \langle \Phi^a, 1_\Omega \rangle \\ &= \Theta_a \sum_{x \in \Omega_a} \sigma_x \\ &= a^{15/8} \sum_{x \in \Omega_a} \sigma_x \,. \end{aligned}$$

Exactly as in Corollary 1.4, m^a_Ω converges in law as $a \searrow 0$. (The limiting law will depend on the boundary condition ξ).

1.2 Choice of the scaling factor

For any bounded domain Ω, the magnetization $M^a := \sum_{x \in \Omega_a} \sigma_x$ has variance

$$\mathrm{Var}\big[M^a\big] = \sum_{x,y \in \Omega_a^2} \mathbb{E}\big[\sigma_x\, \sigma_y\big] \,.$$

It can be shown (see Proposition A.1 in Appendix A) that [3] as $a \searrow 0$,

$$\mathrm{Var}\big[M^a\big] \asymp a^{-4} \mathbb{E}_{\mathbb{C}_a}\big[\sigma_{0_a} \sigma_{(\sqrt{2}+\sqrt{2}i)_a}\big] \,,$$

[3]In this paper $f(a) \asymp g(a)$ as $a \searrow 0$ means that $f(a)/g(a)$ is bounded away from 0 and ∞ while $f(a) \sim g(a)$ means that $f(a)/g(a) \to 1$ as $a \searrow 0$.

where $\mathbb{C}_a$ denotes the square grid $a\mathbb{Z}^2$ and 0_a and $(\sqrt{2}+\sqrt{2}i)_a$ stand for approximations of the points $0, \sqrt{2}+\sqrt{2}i \in \mathbb{C}$.

Following the notation of [CHI12], let us introduce the quantity

$$\varrho\,(a) := \mathbb{E}_{\mathbb{C}_a}[\sigma_{0_a}\sigma_{(\sqrt{2}+\sqrt{2}i)_a}]\,. \tag{1.1}$$

From the above discussion, it is thus natural to scale our magnetization field by a scaling factor of order $a^2\,\varrho\,(a)^{-1/2}$. In most of the rest of this paper (until Section 6), we will assume the following celebrated result by T. T. Wu.

Theorem 1.6 (T. T. Wu, see [Wu66, MW73])**.** *There exists an explicit constant $c > 0$ such that as $a \searrow 0$*

$$\varrho\,(a) \sim c\,a^{1/4}\,. \tag{1.2}$$

Definition 1.7. *Assuming this asymptotic result, we will thus scale our magnetization field by the scaling factor*

$$\Theta_a := a^{15/8}\,. \tag{1.3}$$

Remark 1.8. We believe that it is reasonable to assume Wu's result since it is considered to be among the rigorous results obtained in the theoretical physics literature (yet, according to some experts, although there is no theoretical gap, some details need to be filled in). Nevertheless, this choice may be debatable. In [CHI12], for example, the authors decided to state their result without assuming Wu's result. In our case, if one avoids assuming Wu's asymptotic, all our results remain valid by replacing the above formula for Θ_a by the more cumbersome $\Theta_a := a^2\,\varrho\,(a)^{-1/2}$—see Section 6.

1.3 Brief outline of the proofs

We will give two proofs of our main result, Theorem 1.3, each proof having its own advantage. Let us briefly sketch in this subsection our two strategies. They both start with the same tightness step.

1.3.1 Tightness

The main difficulty here is to find an appropriate functional setup in which the random distribution Φ^a is tight as $a \searrow 0$. We prove in Section 2 that Φ^a is tight in the Sobolev space of negative index $\mathcal{H}^{-3}$. Note that the particular case of the tightness of the random variable $m^a := \langle \Phi^a, 1\rangle$ was already proved in [CN09], see also [C12].

1.3.2 First proof

In the first proof (Section 3), we rely on the FK representation of the Ising model which allows us to decompose the distribution Φ^a as a sum over the FK clusters, where each cluster C carries an independent coin flip $\sigma_C \in \{-1, 1\}$. This approach is convenient since it reintroduces a lot of independence into Φ^a. It also gives an interesting way to think of the limiting field $\Phi^\infty = \lim_{a\to 0}\Phi^a$. The drawback of this approach is that we need to rely on Assumption 3.2. Note that the main argument which consists in constructing "area measures" on critical FK clusters, is somewhat close to the construction of "pivotal measures" in [GPS10].

1.3.3 Second proof

Our second proof (Section 4), as opposed to the first one, does not rely on any assumption (besides assuming Wu's result if one wants to keep the scaling $\Theta_a = a^{15/8}$). For any bounded domain Ω, the idea is to characterize the limit of Φ^a by showing that the quantities

$$\phi_{\Phi^a}(f) = \mathbb{E}\left[e^{\langle \Phi^a, f\rangle}\right],$$

converge as $a \searrow 0$ for any test function $f \in \mathcal{H}^3$. The main ingredients are the breakthrough results by Chelkak, Hongler and Izyurov on the convergence of the k-point correlation functions as well as our Propositions 4.5 and 4.9.

2 Tightness of the magnetization field

Let us first introduce what the setup is when the field Φ^a is defined on the compact square $[0,1]^2$. The extension to general domains as well as to the *full* plane will be given in Subsection 2.2.

2.1 Tightness for Φ^a in a well chosen Sobolev space (case of the square)

In this subsection, we follow (almost word for word) the functional approach which was used by Julien Dubédat in [D09] for another well known field: the Gaussian Free Field.

Let $\mathcal{H}_0^1 = \mathcal{H}_0^1([0,1]^2)$ be the classical Sobolev Hilbert space, i.e., the closure of $C_0^\infty([0,1]^2)$ for the norm

$$\|f\|_{\mathcal{H}^1}^2 := \int_{[0,1]^2} \|\nabla f\|^2 dA\,.$$

Let $\mathcal{H}^{-1}$ be the dual space of $\mathcal{H}_0^1$. It is a space of distributions (i.e., $\mathcal{H}^{-1} \subset D'$) and it is also a Hilbert space equipped with the norm (the *operator norm* on $\mathcal{H}^{-1}$)

$$\|h\|_{\mathcal{H}^{-1}} := \sup_{g \in C_0^\infty([0,1]^2)\,:\,\|g\|_{\mathcal{H}^1} \le 1} \langle h, g\rangle\,.$$

(Here $\langle h, g\rangle$ stands for the evaluation of the distribution h against the test function g).

It will be useful to work with the following basis of $C_0^\infty([0,1]^2)$: for any $j,k \in \mathbb{N}^+$, let

$$e_{j,k}(x,y) := 2\,\sin(j\pi x)\,\sin(k\pi y)\,. \tag{2.1}$$

It is straightforward to check that

$$\begin{cases} (e_{j,k})_{j,k>0} \text{ is a joint othogonal basis for } \mathcal{H}^{-1} \text{ and } \mathcal{H}_0^1 \\ \|e_{j,k}\|_{\mathcal{H}^1}^2 = j^2 + k^2 \\ \|e_{j,k}\|_{\mathcal{H}^{-1}}^2 = \frac{1}{j^2+k^2} \end{cases} \tag{2.2}$$

In particular, if $h = \sum_{j,k} a_{j,k}\, e_{j,k}$, then $\|h\|_{\mathcal{H}^{-1}}^2 = \sum_{j,k} \frac{a_{j,k}^2}{j^2+k^2}$.

More generally, for any $\alpha > 0$, one can define the Hilbert space $\mathcal{H}_0^\alpha$ as the closure of $C_0^\infty([0,1]^2)$ for the norm

$$\|f\|_{\mathcal{H}^\alpha}^2 := \sum_{j,k>0} a_{j,k}^2 (j^2+k^2)^\alpha\,,$$

where $f \in C_0^\infty$ is decomposed as $f = \sum_{j,k>0} a_{j,k}\, e_{j,k}$. Let $\mathcal{H}^{-\alpha}$ be its dual space. It is a Hilbert space with norm

$$\|h\|_{\mathcal{H}^{-\alpha}} := \sup_{g \in C_0^\infty([0,1]^2)\,:\, \|g\|_{\mathcal{H}^\alpha} \le 1} \langle h, g \rangle \,.$$

Furthermore, if $h \in L^2 \subset \mathcal{H}^{-\alpha}$, then h has a Fourier expansion and its $\|\cdot\|_{\mathcal{H}^{-\alpha}}$ norm can be expressed as

$$\|h\|^2_{\mathcal{H}^{-\alpha}} = \sum_{j,k} a^2_{j,k} \frac{1}{(j^2+k^2)^\alpha} \,. \tag{2.3}$$

From now on, for any $a > 0$, we will consider our magnetization field Φ^a as an element of the Polish space $(\mathcal{H}^{-3}, \|\cdot\|_{\mathcal{H}^{-3}})$. Furthermore, since Dirac point masses do not belong to $\mathcal{H}^{-\alpha}$ for $\alpha \le 1/2$, it will be more convenient to change slightly the definition of the distribution Φ^a to the following definition.

Definition 2.1. *We let*

$$\Phi^a := a^{15/8} \sum_{x \in [0,1]^2 \cap a\mathbb{Z}^2} \frac{\sigma_x}{a^2} 1_{S_a(x)} \,,$$

where $S_a(x)$ denotes the square centered at x of side-length a.

As such Φ^a now belongs to L^2 and hence has a Fourier expansion allowing us to rely on formula (2.3) in order to compute $\|\Phi^a\|_{\mathcal{H}^{-\alpha}}$.

Before proving that the random variable Φ^a has a unique scaling limit, a first natural step is to prove some kind of **tightness** for the sequence of random variables $(\Phi^a)_{a>0}$. With this perspective in mind, the following well-known result will be useful to us.

Proposition 2.2 (Rellich Theorem). *For any $\alpha_1 < \alpha_2$, $\mathcal{H}^{-\alpha_1}$ is compactly embedded in $\mathcal{H}^{-\alpha_2}$ ($\mathcal{H}^{-\alpha_1} \subset\subset \mathcal{H}^{-\alpha_2}$). In particular for any $R > 0$, the ball*

$$\overline{B_{\mathcal{H}^{-2}}(0,R)}$$

is compact in $\mathcal{H}^{-3}$.

Thanks to this property, in order to prove tightness, it is enough for us to prove the following result.

Proposition 2.3. *Let us fix some boundary condition ξ on the square $[0,1]^2$. Assume that the boundary condition ξ is made of finitely many arcs of $+,-$ or free type. By Φ^a, we denote the magnetization field within $[0,1]^2 \cap a\mathbb{Z}^2$ subject to the boundary condition ξ. Then as $a \searrow 0$, one has*

$$\limsup_{a \searrow 0} \mathbb{E}\big[\|\Phi^a\|^2_{\mathcal{H}^{-2}}\big] < \infty \,,$$

uniformly in the boundary conditions ξ and thus $(\Phi^a)_{a>0}$ is **tight** *in the space $\mathcal{H}^{-3}$.*

Proof. We wish to bound from above the quantity

$$\begin{aligned}\mathbb{E}\big[\|\Phi^a\|^2_{\mathcal{H}^{-2}}\big] &= \mathbb{E}\big[\sum_{j,k>0} \langle \Phi^a, e_{j,k}\rangle^2 \frac{1}{(j^2+k^2)^2}\big] \\ &= \sum_{j,k>0} \frac{1}{(j^2+k^2)^2} \mathbb{E}\big[\langle \Phi^a, e_{j,k}\rangle^2\big]\,.\end{aligned}$$

The following lemma concludes the proof of Proposition 2.3:

Lemma 2.4. *There is a constant $C>0$ such that for all $j,k>0$*

$$\limsup_{a\to 0}\sup_{j,k} \mathbb{E}\big[\langle \Phi^a, e_{j,k}\rangle^2\big] < C\,.$$

Proof.

$$\begin{aligned}\mathbb{E}\big[\langle \Phi^a, e_{j,k}\rangle^2\big] \leq & a^{15/4} \sum_{x\neq y\in[0,1]^2\cap a\mathbb{Z}^2} \Big| \iint_{S_a(x)\times S_a(y)} \frac{\mathbb{E}\big[\sigma_x\,\sigma_y\big]}{a^4} e_{j,k}(\bar x) e_{j,k}(\bar y) dA(\bar x) dA(\bar y)\Big| \\ & + a^{15/4} \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} \Big(\int_{S_a(x)} \frac{1}{a^2} e_{j,k}(x) dA(\bar x)\Big)^2 . \\ \leq & a^{15/4} \|e_{j,k}\|^2_\infty \sum_{x\neq y\in[0,1]^2\cap a\mathbb{Z}^2} |\mathbb{E}\big[\sigma_x\,\sigma_y\big]| + a^{15/4}\|e_{j,k}\|^2_\infty \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} 1 \\ \leq & O(1)\,,\end{aligned}$$

uniformly in j,k and the boundary condition ξ (indeed, by FKG for the FK representation, it is enough to dominate $\mathbb{E}\big[\sigma_x\sigma_y\big]$ by the extreme boundary conditions $\xi=+$ or $\xi=-$). □

Remark 2.5. Using Lemma 2.4 as is, it is straightforward to strengthen the above proposition by showing that for any $\epsilon>0$, $(\Phi^a)_{a>0}$ is in fact **tight** in the space $\mathcal{H}^{-1-\epsilon}$. It is thus natural to wonder for which values of $\alpha>0$, $(\Phi^a)_{a>0}$ remains tight in $\mathcal{H}^{-\alpha}$? It is clear that there is a lot of room if one wishes to obtain better estimates than the one provided by Lemma 2.4. Yet it appears that there is some $\bar\alpha>0$ such that $(\Phi^a)_{a>0}$ is not tight in $\mathcal{H}^{-\alpha}$ when $\alpha<\bar\alpha$. In particular, it appears that $\Phi=\lim\Phi^a$ is less regular than the planar Gaussian Free Field.

2.2 Extension to other domains and to the full plane

For the extension to other planar domains, it is tempting to use the conformal invariance of $\|\cdot\|_{\mathcal{H}^1_0}$. I.e., if $\Omega \subsetneqq \mathbb{C}$ is some simply connected domain of the plane, and if $\phi:\mathbb{D}\to\Omega$ is a conformal map from the disc to the domain Ω, then for any $f:\Omega\to\mathbb{C}$, one has

$$\begin{aligned}\|f\|^2_{\mathcal{H}^1_0(\Omega)} &= \int_\Omega \|\nabla f\|^2 dA \\ &= \int_D \|\nabla(f\circ\phi)\|^2 dA \\ &= \|f\circ\phi\|^2_{\mathcal{H}^1_0(\mathbb{D})}\,.\end{aligned}$$

Unfortunately this is of no use here since at the discrete level, $a\,\mathbb{Z}^2$ is not conformally invariant, and furthermore we did not prove that (Φ^a) was tight in the dual space $\mathcal{H}^{-1}$.

Hence, one needs to rely on a slightly more hands-on approach. The difficulty to overcome here is that for a generic domain Ω, one does not have an orthogonal basis for the Laplacian with bounded L^∞ norm. Yet, as we will see below, one can still use the fact that Proposition 2.3 is uniform in the boundary conditions ξ.

Let us first tackle the case of bounded domains.

2.2.1 Case where $\Omega \subsetneq \mathbb{C}$ is a bounded simply connected domain of the plane, with prescribed boundary condition ξ on $\partial\Omega$

Let $(Q_i)_{i\in\mathbb{N}}$ be a *Whitney decomposition* of Ω into disjoint squares. For any $a>0$ let Φ^a_Ω be the magnetization field on $\Omega\cap a\mathbb{Z}^2$ induced by the boundary condition ξ. One can write Φ^a as

$$\Phi^a = \sum_{i\in\mathbb{N}} \Phi^a_{|Q_i}\,.$$

By the triangle inequality, one has that

$$\|\Phi^a\|_{\mathcal{H}^{-2}} \le \sum_{i\in\mathbb{N}} \|\Phi^a_{|Q_i}\|_{\mathcal{H}^{-2}}\,.$$

Now the key step is to notice that the proof of Proposition 2.3 immediately gives

Lemma 2.6. *There exists a uniform constant $C>0$ such that for any domain Ω and any boundary condition ξ on $\partial\Omega$, if Q_i is a square inside Ω (with area $\lambda(Q_i)$), then for any $a>0$:*

$$\mathbb{E}\big[\|\Phi^a_{\big|Q_i}\|^2_{\mathcal{H}^{-2}}\big] \le C\lambda(Q_i)^{15/8}\,. \tag{2.4}$$

Proof (sketch). To see why this holds, take a square Q_i inside Ω. Let q be its side-length so that $q^2=\lambda(Q_i)$. By renormalizing the scale by a factor $1/q$, one can see that our field $\Phi^a_{|Q_i}$ has the same $\mathcal{H}^{-2}$ norm as

$$q^{15/8}\times \Phi^{a/q}_{|\frac{1}{q}Q_i}\,.$$

But now, $\frac{1}{q}Q_i$ is a square of side-length 1, therefore by Proposition 2.3 (which was uniform in the outer boundary condition)

$$\mathbb{E}\big[\|\Phi^{a/q}_{\big|\frac{1}{q}Q_i}\|^2_{\mathcal{H}^{-2}}\big] \le C.$$

This gives

$$\mathbb{E}\big[\|\Phi^a_{\big|Q_i}\|^2_{\mathcal{H}^{-2}}\big] \le q^{15/4}C = C\lambda(Q_i)^{15/8}\,.$$

□

By Cauchy-Schwarz, this implies that

$$\mathbb{E}\big[\|\Phi^a_{\big|Q_i}\|_{\mathcal{H}^{-2}}\big] \le C^{1/2}\lambda(Q_i)^{15/16}\,. \tag{2.5}$$

From this formula, one can see that one cannot hope to prove a tightness result for Φ^a on the full domain Ω. Indeed there are bounded domains for which $\sum_i \lambda(Q_i)^{15/16}$ diverges. Yet, for our purposes, it will be sufficient to prove the following weaker result.

Proposition 2.7. *Let Ω be a bounded simply connected domain of the plane. For any open set U whose closure $\overline{U}$ is contained inside Ω, there is a constant $C = C_U > 0$ such that for any boundary condition ξ on $\partial\Omega$, one has*

$$\mathbb{E}\big[\|\Phi^a_{|U}\|_{\mathcal{H}^{-2}}\big] < C_U \,.$$

Hence the restriction of $(\Phi^a)_{a>0}$ to the open subset U is a tight sequence in $\mathcal{H}^{-3}$.

Proof. Observe that

$$\mathbb{E}\big[\|\Phi^a_{|U}\|_{\mathcal{H}^{-2}}\big] \le C^{1/2} \sum_{i, Q_i \cap U \neq \emptyset} \lambda(Q_i)^{15/16} \,.$$

By the properties of Whitney decompositions, only finitely many Q_i intersect the subset U, hence the above sum is finite and is bounded from above by some constant $C = C(U) > 0$. □

2.2.2 Case of the infinite plane

(The case of non-bounded simply connected domains is treated similarly.)

Our magnetization field $\Phi^a := \sum_{x \in a\mathbb{Z}^2} a^{15/8}\, \sigma_x \delta_x$ is well defined as a distribution on the full plane $\mathbb{R}^2$. One natural way to proceed in order to keep some tightness is to view our field as a nested sequence of restricted fields: $(\Phi^a_{|B_k})_{k\ge 1}$ where B_k is the square $[-2^k, 2^k]^2$. This sequence of nested distributions lives in the product of Hilbert spaces

$$\mathcal{H}^{-3}_\infty := \prod_{k\ge 1} \mathcal{H}^{-3}_{B_k} \,,$$

where for each $k \ge 1$, $\mathcal{H}^{-3}_{B_k}$ denotes the dual of $\mathcal{H}^3_0(B_k)$.

Since for any $k \ge 1$, $(\mathbb{E}\big[\|\Phi^a_{|B_k}\|_{\mathcal{H}^{-2}_{B_k}}\big])_{a>0}$ is a bounded sequence (by $O(2^{15k/8})$), the sequence of random variables $(\Phi^a_{|B_k})_{a>0}$ is tight in the space $\mathcal{H}^{-3}_{B_k}$. In particular, there is a subsequential scaling limit, i.e., there is a random field $\Phi_k \in \mathcal{H}^{-3}_{B_k}$ and a sequence $(a^k_m)_{m\ge 1}$ with $a^k_m \searrow 0$ such that

$$\Phi^{a^k_m}_{|B_k} \xrightarrow[m\to\infty]{d} \Phi_k \,,$$

in law (for the topology on $\mathcal{H}^{-3}_{B_k}$ induced by $\|\cdot\|_{\mathcal{H}^{-3}}$). Furthermore, from k to $k+1$, one can choose the subsequential scaling limit $(a^{k+1}_m)_{m\ge 1}$ so that $\{a^{k+1}_m\}_m \subset \{a^k_m\}_m$. This allows us to define a "joint" subsequential scaling limit along the sequence

$$\bar{a}_m := a^m_m \,.$$

Doing so, the sequence $(\Phi^{\bar{a}_m}_{|B_k})_{k\ge 1}$ converges in law (for the product topology) to

$$(\Phi_k)_{k\ge 1} \in \mathcal{H}^{-3}_\infty \,.$$

It is obvious (going back to the discrete mesh fields $\Phi^a_{|B_k}$) that a.s. for any $k \ge 1$, one has

$$\Phi_{k+1} 1_{B_k} \equiv \Phi_k \,.$$

2.3 Subsequential scaling limit

Proposition 2.3 has the following immediate corollary.

Corollary 2.8. *Let ξ be a boundary condition on the square $[0,1]^2$ as in Proposition 2.3, then there is a subsequential scaling limit Φ^*, i.e., a random distribution $\Phi^* \in \mathcal{H}^{-3}$ such that for a certain subsequence $a_k \searrow 0$, Φ^{a_k} converges in law to Φ^* for the topology on $\mathcal{H}^{-3}$ induced by $\|\cdot\|_{\mathcal{H}^{-3}}$.*

In the rest of the paper, in order to simplify the notation, we will stick to the case of the square $[0,1]^2$. The extension to other domains as well as to the full plane can be done using the methods that have been presented in this section.

3 First proof of the scaling limit of Φ^a using area measures on FK clusters

3.1 Setup for the proof of convergence

The proof will proceed as follows. The main idea of this proof of Theorem 1.2 is to rely on the recent breakthrough results by Smirnov [Sm10]. Our goal will be to show that the scaling limit Φ^∞ of the magnetization field Φ^a is *measurable* with respect to an appropriate scaling limit of FK-Ising percolation. Yet, we will need to rely on Assumption 3.2 below, which can be seen as a slight extension of the results announced in [KS].

3.1.1 Notation, space of percolation configurations, compactness

We will work with the following setup : denote by σ_a the critical Ising configuration on $a\mathbb{Z}^2 \cap [0,1]^2$. As is well known, the configuration σ_a can be obtained from an FK configuration ω_a on $a\mathbb{Z}^2 \cap [0,1]^2$ by flipping an independent $\{\pm\}$ fair coin for each cluster of ω_a. Let ω_a^+ (resp. ω_a^-) be the configuration consisting of the clusters of ω_a which have been chosen to be plus (resp., minus). Let us denote by $\bar\omega_a$ the coupled pair (ω_a^+, ω_a^-). Note that one has $\omega_a = \omega_a^+ \cup \omega_a^-$.

It will be very convenient to consider these FK configurations ω_a as (random) variables in a compact metrizable space $(\mathscr{H}, \mathcal{T}) = (\mathscr{H}_{[0,1]^2}, \mathcal{T})$ which encodes all macroscopic crossing events. This compact space is not specific to our study of FK percolation and one can in fact rely here on the setup which was introduced by Schramm and Smirnov in [ScSm11] in the case of independent percolation ($q = 1$). Very briefly, it works as follows: the space of percolation configurations built in [ScSm11] is the space of **closed hereditary subsets** of the space of *quads* $(\mathcal{Q}, d_\mathcal{Q})$. Roughly speaking, this means that a point $\omega \in \mathscr{H}$ corresponds to a family of quads $Q \in \mathcal{Q}$ which is closed in $(\mathcal{Q}, d_\mathcal{Q})$ and which satisfies the following constraint: if $Q \in \omega$ and Q' is "easier" to be traversed, then Q' is in ω as well. In [ScSm11], it is proved that this space $\mathscr{H}$ can be endowed with a topology $\mathcal{T}$ so that the topological space $(\mathscr{H}, \mathcal{T})$ is compact, Hausdorff and metrizable. For convenience, we will choose a (non-explicit) metric $d_\mathscr{H}$ on $\mathscr{H}$ which induces the topology $\mathcal{T}$. See [ScSm11] for a clear exposition of the topological space $(\mathscr{H}, \mathcal{T})$. See also [GPS10, GPS].

Since we will need the crossing properties of the + versus the - clusters, we will in fact consider $\bar\omega_a = (\omega_a^+, \omega_a^-)$ as a random variable in the compact metrizable space $\mathscr{H} \times \mathscr{H}$ endowed with the product topology.

What is known about the limit as $a \to 0$ of the coupling $\bar{\omega}_a = (\omega_a^+, \omega_a^-) \in \mathscr{H} \times \mathscr{H}$? First of all, the tightness for $(\bar{\omega}_a)_{a>0}$ follows immediately from the compactness of $(\mathscr{H} \times \mathscr{H}, \mathcal{T} \otimes \mathcal{T})$:

Fact 3.1. *The random variable $\bar{\omega}_a = (\omega_a^+, \omega_a^-)$ is in $\mathscr{H} \times \mathscr{H}$ (with the product topology). Since $\mathscr{H} \times \mathscr{H}$ is compact for the product topology $\mathcal{T} \otimes \mathcal{T}$, there are subsequential scaling limits for $(\bar{\omega}_a)_{a>0}$ as $a \to 0$.*

It now remains to identify uniquely the possible subsequential scaling limits.

3.1.2 Scaling limit for $(\bar{\omega}_a)_{a>0}$

It is known since the breakthrough paper [Sm10] that certain discrete "observables" for critical FK-percolation are asymptotically *conformally invariant*. These observables can then be used ([Sm]) to prove that interfaces have a scaling limit described by $SLE_{16/3}$ curves. In our case, we need a *full scaling limit* result. Indeed, our later results in this section of the paper are based on the following hypothesis.

Assumption 3.2. The coupled configurations $\bar{\omega}_a = (\omega_a^+, \omega_a^-)$ considered as random points in $(\mathscr{H} \times \mathscr{H},\ \mathcal{T} \otimes \mathcal{T})$ have a (unique) scaling limit as $a \searrow 0$; they converge in law to a continuum FK $\bar{\omega}_\infty = (\omega_\infty^+, \omega_\infty^-)$.

This assumption is very reasonable, based on the convergence of discrete interfaces to $SLE_{16/3}$ curves which will appear in [Sm]. An even clearer evidence is provided by the work in progress [KS], where it is shown that the branching exploration tree converges to the branching $SLE_{16/3}$ tree. However, as explained in [ScSm11], it is not always easy to go from one notion of scaling limit to another. In the case of standard percolation ($q = 1$), the first and third author proved ([CN06]) the existence and several properties of the full scaling limit as the collection of all interfaces, building the limit object from SLE_6 loops, and, as explained in [GPS10, Section 2.3], their results imply convergence also in the "quad topology" $(\mathscr{H}, \mathcal{T})$.

In our present case, from the convergence of ω_a to $CLE_{16/3}$ (as announced in [KS]), it is not hard to represent the scaling limit ω_∞ as a countable family of nested loops as in [CN06]. Now, following [CN06] (as explained in [GPS10, Section 2.3]), plus the fact that the 6-arm exponent for critical FK percolation is also > 2 (see [CDH], Corollary 5.9) one should be able to obtain in the same fashion the convergence of ω_a to ω_∞ in the sense of the topological space $(\mathscr{H}, \mathcal{T})$.

This step would justify the convergence of ω_a to ω_∞ in $(\mathscr{H}, \mathcal{T})$, but we need slightly more, i.e., convergence of $\bar{\omega}_a$ to $\bar{\omega}_\infty$. This is not the hard part of the story. Indeed, from the $CLE_{16/3}$ exploration tree, we argued above that we can see ω_∞ as a countable collection of nested loops (as in [CN06]). The collection of countably many *clusters* is measurable with respect to this notion of scaling limit. One can thus "toss a coin" for each of them and obtain two configurations $\omega_\infty^\pm$, each of them being seen as a countable (sub-)family of nested loops. There are no 6-arm events either in these configurations, and one can thus apply the same arguments (from [CN06]) to give a limiting $(\omega_\infty^+, \omega_\infty^-) \in \mathscr{H} \times \mathscr{H}$. Note that the preceding discussion is an explanation of why Assumption 3.2 is reasonable but it is not a complete proof.

3.1.3 Measurable events in $\mathscr{H} \times \mathscr{H}$

In this subsection, we follow very closely Subsection 2.4. in [GPS10]. We refer to that paper for more details and will only highlight briefly how to adapt the definitions to our present case.

Let $A = (\partial_1 A, \partial_2 A)$ be a fixed topological piecewise smooth annulus embedded in our domain $[0,1]^2$. We will often rely on the events $\mathcal{A}^{\pm} = \mathcal{A}_1^{\pm}$ which are in the Borel sigma field of $(\mathscr{H} \times \mathscr{H}, \mathcal{T} \otimes \mathcal{T})$ and which are defined as follows:

$$\mathcal{A}_1^{+} := \left\{ \bar\omega \in \mathscr{H} \times \mathscr{H},\, \exists Q \in \mathcal{Q} \text{ s.t. } Q \in \omega^{+} \text{ and } Q \text{ connects } \partial_1 A \text{ with } \partial_2 A \right\} . \tag{3.1}$$

The event $\mathcal{A}_1^{-}$ is defined in the obvious related manner. We may also define the one-arm event $\mathcal{A}_1$ on the "uncolored" space $\mathscr{H}$. We will need the following extension of Lemma 2.4 in [GPS10] whose proof applies easily to our present case, assuming the 5-arm and the bound on the 6-arm exponent for FK percolation obtained in [CDH], Theorem 5.8 and Corollary 5.9.

Lemma 3.3 (see Lemma 2.4 in [GPS10])**.** *Let A be a piecewise smooth annulus in $[0,1]^2$. Then,*

$$\mathbb{P}\big[\bar\omega_a \in \mathcal{A}_1^{\pm}\big] \to \mathbb{P}\big[\bar\omega_\infty \in \mathcal{A}_1^{\pm}\big] \,,$$

as the mesh size $a \to 0$. Furthermore, in any coupling of the measures $\mathbb{P}_a$ and $\mathbb{P}_\infty$ on the space $(\mathscr{H} \times \mathscr{H}, \mathcal{T} \otimes \mathcal{T})$, in which $\bar\omega_a \to \bar\omega_\infty$ a.s. we have that $1\!\!1_{\mathcal{A}_1^{\pm}}(\bar\omega_a) \to 1\!\!1_{\mathcal{A}_1^{\pm}}(\bar\omega_\infty)$ almost surely.

3.1.4 General setup of convergence : the space $\mathscr{H} \times \mathscr{H} \times \mathcal{H}^{-3}$

Let us consider the coupling $(\bar\omega_a, \Phi^a(\bar\omega_a)) = (\omega_a^{+}, \omega_a^{-}, \Phi^a) \in \mathscr{H} \times \mathscr{H} \times \mathcal{H}^{-3}$. In order to prove our main Theorem 1.2, will will prove the following stronger result.

Theorem 3.4 (under Assumption 3.2)**.** *The random variables $(\bar\omega_a, \Phi^a) \in \mathscr{H} \times \mathscr{H} \times \mathcal{H}^{-3}$ converge in law as the mesh size $a \to 0$ to $(\bar\omega_\infty, \Phi^\infty)$ for the topology induced by the metric* [4] *$d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \| \cdot \|_{\mathcal{H}^{-3}}$.*

Furthermore, the limiting random variable $\Phi^\infty \in \mathcal{H}^{-3}$ is measurable with respect to $\bar\omega_\infty$. I.e., we have

$$\Phi^\infty = \Phi^\infty(\bar\omega_\infty) \,.$$

From Proposition 2.3, we already know that $(\bar\omega_a, \Phi^a(\bar\omega))$ is tight in the space $\mathscr{H} \times \mathscr{H} \times \mathcal{H}^{-3}$ endowed with the metric $d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \| \cdot \|_{\mathcal{H}^{-3}}$. As in Corollary 2.8, one thus has subsequential scaling limits: i.e., one can find a subsequence $a_k \to 0$ such that $(\bar\omega_{a_k}, \Phi^{a_k})$ converges in law to $(\bar\omega_\infty, \Phi^*)$ (here we use Assumption 3.2 which says that there is a unique possible subsequential scaling limit for $\bar\omega_a$). Since the space $(\mathscr{H} \times \mathscr{H} \times \mathcal{H}^{-3}, d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \| \cdot \|_{\mathcal{H}^{-3}})$ is a complete separable metric space, one can apply Skorohod's Theorem. This gives us a joint coupling of the above processes such that

$$(\bar\omega_{a_k}, \Phi^{a_k}) \xrightarrow{a.s.} (\bar\omega_\infty, \Phi^*) \,. \tag{3.2}$$

Proving Theorem 3.4 boils down to proving that Φ^* is in fact measurable with respect to $\bar\omega_\infty$. Achieving this would indeed conclude the proof of Theorem 3.4 (and thus Theorem 1.2) since it would uniquely characterize the subsequential scaling limits of $(\bar\omega_a, \Phi^a)$.

The purpose of the next subsection is to reduce the proof of Theorem 3.4 to the study of the renormalized magnetization in a dyadic box.

[4] Recall from Subsection 3.1.1 that we have chosen a metric $d_{\mathscr{H}}$ on $\mathscr{H}$ which induces the topology $\mathcal{T}$.

3.1.5 Reduction to the renormalized magnetization in a dyadic box

We wish to prove Theorem 3.4, i.e., to show that if $(\bar{\omega}_{a_k}, \Phi^{a_k}) \xrightarrow{a.s.} (\bar{\omega}_\infty, \Phi^*)$, then Φ^* can be expressed as a measurable function of $\bar{\omega}_\infty$. Since $\Phi^* \in \mathcal{H}^{-3}$, it can be decomposed in the orthonormal basis of L^2, $(e_{j,k})_{j,k\geq 1}$, that we already used in Section 2:

$$\Phi^* = \sum_{j,k\geq 1} \langle \Phi^*, e_{j,k}\rangle \, e_{j,k} \,. \tag{3.3}$$

From the a.s. convergence (3.2), we have that for any fixed $j,k \geq 1$:

$$(\bar{\omega}_{a_k}, \langle \Phi^{a_k}, e_{j,k}\rangle) \xrightarrow{a.s.} (\bar{\omega}_\infty, \langle \Phi^*, e_{j,k}\rangle)\,,$$

where the convergence holds for the metric $d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \|\cdot\|_{\mathbb{R}}$. In order to prove Theorem 3.4, thanks to the decomposition (3.3), it only remains to to prove that for each fixed $j,k \geq 1$, the limiting quantity $\langle \Phi^*, e_{j,k}\rangle$ is itself measurable w.r.t. $\bar{\omega}_\infty$.

It turns out that one can further reduce the difficulty of this task by approximating the function $e_{j,k}$ using Riemann integrals as follows. Let us fix some $j,k \geq 1$. For any small $\beta > 0$, one can find dyadic squares B_i and real numbers b_i so that if $g_\beta := \sum_i b_i 1_{B_i}$, then

$$\|e_{j,k} - g_\beta\|_{L^\infty([0,1]^2)} < \beta\,.$$

Now, exactly as in the proof of Lemma 2.4, it is not hard to check that

$$\mathbb{E}\big[\big(\langle \Phi^a, e_{j,k}\rangle - \langle \Phi^a, g_\beta\rangle\big)^2\big] \leq C\,\beta^2\,,$$

uniformly in $a > 0$ (for some universal constant $C > 0$). If one can show that, as $a_k \searrow 0$, $\langle \Phi^{a_k}, g_\beta\rangle$ converges a.s. to a measurable function $G_\beta(\bar{\omega}_\infty)$, then using the uniform L^2 bounds from Section 2 together with the triangle inequality in L^2, it follows that $G_\beta(\bar{\omega}_\infty)$ converges as $\beta \to 0$ in L^2 to $\langle \Phi^*, e_{j,k}\rangle$. Since L^2 is complete, $G_\beta(\bar{\omega}_\infty)$ has an L^2-limit G_0 as $\beta \to 0$ which is itself measurable w.r.t. $\bar{\omega}_\infty$ and one has necessarily that $\langle \Phi^*, e_{j,k}\rangle \overset{a.s.}{=} G_0(\bar{\omega}_\infty)$. Since G_β is a linear combination of magnetizations in dyadic squares, it follows from the above discussion that Theorem 3.4 is a corollary of the following theorem.

Theorem 3.5. *Let B be any dyadic square in $[0,1]^2$ and let the renormalized magnetization in B be the random variable*

$$m_B^a = m^a := a^{15/8} \sum_{x\in a\mathbb{Z}^2\cap B} \sigma_x\,.$$

Then the coupled random variable $(\bar{\omega}_a, m^a) \in \mathscr{H} \times \mathscr{H} \times L^2$ converges in law as the mesh size $a \to 0$ to $(\bar{\omega}_\infty, m)$ for the topology induced by the metric $d_{\mathscr{H}} \times d_{\mathscr{H}} \times \|\cdot\|_{L^2}$. Furthermore, the limiting random variable $m \in L^2$ is measurable with respect to $\bar{\omega}_\infty$. I.e., we have

$$m = m(\bar{\omega}_\infty)\,.$$

We now turn to the proof of this theorem. Without loss of generality and for the sake of simplicity, we may assume that our dyadic square B is just $[0,1]^2$.

3.2 Scaling limit for the magnetization random variable (proof of Theorem 3.5)

3.2.1 Structure of the proof of Theorem 3.5

The setup for the scaling limit of m^a is similar to the setup we explained above (in Subsection 3.1.4) for the scaling limit of Φ^a. Namely, we consider the coupling $(\bar\omega_a, m^a)$ embedded in the metric space $(\mathscr{H}^2 \times L^2, d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \|\cdot\|_{L^2})$. The tightness of $(\bar\omega_a, m^a)$ easily follows from the stronger tightness of Proposition 2.3 (see also [CN09] and [C12]). In particular, there exist subsequential scaling limits

$$(\bar\omega_{a_k}, m^{a_k}) \xrightarrow{d} (\bar\omega^*_\infty, m^*)\,.$$

By Assumption 3.2, there is a unique possible law for $\bar\omega^*_\infty$, which we denoted by $\bar\omega_\infty$. In order to prove Theorem 3.5, it remains to show that the second coordinate $m^* \in L^2$ is measurable with respect to the first one.

As previously, let us couple all these random variables using Skorohod's Theorem so that

$$(\bar\omega_{a_k}, m^{a_k}) \xrightarrow{a.s.} (\bar\omega_\infty, m^*)\,, \tag{3.4}$$

for the metric $d_{\mathscr{H}} \oplus d_{\mathscr{H}} \oplus \|\cdot\|_{L^2}$.

The main idea will be to approximate the quantity m^a by relying only on "macroscopic information" from the coupled configuration $\bar\omega_a$. The "macroscopic quantities" we are allowed to use are the quantities which are preserved in the scaling limit $\bar\omega_a \to \bar\omega_\infty$ (i.e., crossing events and so on, see Subsection 3.1.3).

We will approximate the magnetization m^a by a two step procedure. First, let us fix some small dyadic scale $\rho \in \{2^{-k}, k \in \mathbb{N}\}$. Divide the square $[0,1]^2$ along the grid $\rho\mathbb{Z}^2$. Let S_ρ be the set of ρ-squares thus obtained. For each ρ-square $Q \in S_\rho$, consider the annulus $A_Q := 3Q \setminus Q$ where we denote by $3Q$ the square of side-length 3ρ centered on Q. We will divide the clusters in the FK-configuration ω_a in two groups: the clusters which cross at least one annulus $A_Q, Q \in S_\rho$ and the clusters which do not cross any annulus. We may rewrite the magnetization m^a as follows:

$$\begin{aligned} m^a &= \sum_{x \in [0,1]^2 \cap a\mathbb{Z}^2} a^{15/8}\sigma_x && (3.5)\\ &= \sum_{Q \in S_\rho} \Big(\sum_{x \in Q : x \leftrightarrow \partial(3Q)} a^{15/8}\sigma_x + \sum_{x \in Q : x \not\leftrightarrow \partial(3Q)} a^{15/8}\sigma_x \Big)\,. && (3.6) \end{aligned}$$

Following [CN09] (with a slightly different setup here), let us show that the contribution of the second inside sum is negligible in L^2. Indeed,

$$\begin{aligned} \Big\| \sum_{Q \in S_\rho} \sum_{x \in Q : x \not\leftrightarrow \partial(3Q)} a^{15/8}\sigma_x \Big\|_2^2 &= \sum_{Q,Q'} \sum_{x,y} a^{15/4}\mathbb{E}\big[\sigma_x\sigma_y 1_{x\in Q: x \not\leftrightarrow \partial(3Q)} 1_{y \in Q' : y \not\leftrightarrow \partial(3Q')}\big] \\ &= \sum_{Q,Q'} \sum_{x,y} a^{15/4}\mathbb{E}\big[1_{x \leftrightarrow y} 1_{x\in Q: x \not\leftrightarrow \partial(3Q)} 1_{y \in Q' : y \not\leftrightarrow \partial(3Q')}\big] && (3.7)\\ &\leq \sum_{x,y : |x-y| \leq 8\rho} a^{15/4}\mathbb{E}\big[1_{x\leftrightarrow y}\big] && (3.8)\\ &= a^{15/4} O(a^{-2}(\frac{8\rho}{a})^2 (\rho/a)^{-1/4}) && (3.9)\\ &= O(\rho^{7/4}) && (3.10) \end{aligned}$$

Since we are looking for a limiting law for m^a in L^2, it is thus enough (up to a small error of $O(\rho^{7/4})$) to focus on the first summand

$$m_\rho^a := \sum_{Q\in S_\rho} \sum_{x\in Q: x\leftrightarrow\partial(3Q)} a^{15/8}\sigma_x \,.$$

Since $\rho > 0$ is fixed and the mesh size $a \to 0$, we are getting closer to an approximation by "macroscopic quantities". We still need to approximate in a suitable macroscopic manner the quantity

$$L_Q^a = L_Q^a(\bar{\omega}_a) := \sum_{x\in Q: x\leftrightarrow\partial(3Q)} a^{15/8}\sigma_x \,,$$

for each ρ-square $Q \in S_\rho$. This is the second step of our approximation procedure and for this, we will follow very closely the proof in [GPS10] of the scaling limit of counting measures on pivotal points (called *pivotal measures*). In the rest of the proof, let us fix the value of ρ and fix some ρ-square $Q \in S_\rho$.

Let $\epsilon > 0$ be some small fixed threshold (such that $a \ll \epsilon \ll \rho$). Divide the square $Q \cap a\mathbb{Z}^2$ into equal disjoint squares of side-length $\epsilon_a := a\lfloor \epsilon/a \rfloor$. There are $N = \Omega(\epsilon^{-2})$ such squares inside Q (we don't need to keep the dependence in ρ in what follows) plus $O(\epsilon^{-1})$ squares which intersect the boundary of Q. Let $\big(B_i\big)_{i\in\{1,\ldots,N\}}$ denote the set of such ϵ_a-squares inside Q.

For each $i \in [N] := \{1, \ldots, N\}$, let

$$X_i^\epsilon = X_i := \sum_{x\in B_i: x\leftrightarrow\partial(3Q)} a^{15/8}\sigma_x \,. \tag{3.11}$$

Furthermore, let $B := \bigcup B_i \subset Q$. We thus have

$$L_Q^a = \sum_{i\in[N]} X_i^\epsilon + \sum_{x\in Q\setminus B\,: x\leftrightarrow\partial(3Q)} a^{15/8}\sigma_x \,. \tag{3.12}$$

The second term (which arises when ρ is not a multiple of ϵ_a) turns out to be negligible in L^2 as well. Indeed,

$$\begin{aligned}
\| \sum_{x\in Q\setminus B\,: x\leftrightarrow\partial(3Q)} a^{15/8}\sigma_x \|_2^2 \quad &\le \quad \sum_{x,y\in Q\setminus B} a^{15/4}\mathbb{P}\big[x \leftrightarrow y\big] && (3.13)\\
&\le \quad O(1)\, a^{15/4}\, \frac{\rho}{\epsilon} \sum_{k=1}^{\rho/\epsilon} \big(\frac{\epsilon}{a}\big)^4 \big(\frac{a}{k\epsilon}\big)^{1/4} && (3.14)\\
&\le \quad O(1)\, \frac{\rho}{\epsilon}\, (\rho/\epsilon)^{3/4}\epsilon^4\epsilon^{-1/4} && (3.15)\\
&\le \quad O(1)\, \rho^{7/4}\epsilon^2 \,. && (3.16)
\end{aligned}$$

Therefore as ϵ goes to zero, and uniformly in the mesh size $a \le \epsilon$, the boundary term is negligible in L^2. It thus remains to control the term $\sum_{i\in[N]} X_i^\epsilon$.

For this, let us introduce for each $i \in [N]$, the variables

$$Y_i^\epsilon = Y_i^\epsilon(\bar{\omega}_a) := 1_{\{B_i \overset{\omega_a^+}{\longleftrightarrow} \partial(3Q)\}} - 1_{\{B_i \overset{\omega_a^-}{\longleftrightarrow} \partial(3Q)\}} \,. \tag{3.17}$$

We will prove in the next subsection the following proposition.

Proposition 3.6. *There exists a universal constant $c > 0$ such that for any square Q of side-length ρ as above, we have*

$$\| \sum X_i^\epsilon - c\beta(\epsilon) \sum Y_i^\epsilon \|_2 \longrightarrow 0\,, \tag{3.18}$$

as $\epsilon \to 0$ uniformly in $a \le \epsilon$ and where $\beta(\epsilon) := \epsilon^2 \, \alpha_1^{\mathrm{FK}}(\epsilon, 1)^{-1}$ is the one-arm probability for FK-Ising percolation in the annulus $[-1,1]^2 \setminus [-\epsilon/2, \epsilon/2]^2$.

Before proving the proposition, let us explain why it indeed implies Theorem 3.5. From Subsection 3.1.3, there are measurable functions $Y_i^\epsilon(\bar\omega_\infty)$ such that for each $i \in [N]$ and along the above subsequence (a_k) (see Equation (3.4)):

$$Y_i^\epsilon(\bar\omega_{a_k}) \xrightarrow{a.s.} Y_i^\epsilon(\bar\omega_\infty)\,.$$

Furthermore, one can see from Proposition 3.6 that $\|\beta(\epsilon) \sum Y_i^\epsilon(\bar\omega_a)\|_2$ is bounded uniformly in $0 < a \le \epsilon$. This implies, modulo some triangle inequalities that

$$\|L_Q^a(\bar\omega_{a_k}) - c\beta(\epsilon) \sum Y_i^\epsilon(\bar\omega_\infty)\|_2 \longrightarrow 0\,,$$

uniformly in $0 < a_k < \epsilon$. This in turn implies that the sequence $\Big(c\beta(\epsilon) \sum Y_i^\epsilon(\bar\omega_\infty)\Big)_{\epsilon>0}$ is a Cauchy-sequence in L^2. In particular, it has an L^2-limit that we may denote by $L_Q(\bar\omega_\infty)$ and this L^2-limit is such that

$$\|L_Q^{a_k}(\bar\omega_{a_k}) - L_Q(\bar\omega_\infty)\|_2 \longrightarrow 0\,,$$

as the mesh size $a_k \to 0$.

Using the above estimates, we have that

$$\|m^{a_k} - \sum_Q L_Q(\bar\omega_\infty)\|_2^2 \longrightarrow 0\,,$$

uniformly in $0 < a_k < \rho$. Exactly as above with the second order approximation in ϵ, the above displayed equation (plus the L^2 bounds we already have) implies that the Cauchy sequence $\Big(\sum_Q L_Q(\bar\omega_\infty)\Big)_{\rho>0}$ has an L^2-limit denoted by $m(\bar\omega_\infty)$ as $\rho \to 0$. Finally, thanks to the a.s. convergence in Equation (3.4), this L^2-limit must be such that

$$m^* \overset{a.s.}{=} m(\bar\omega_\infty)\,, \tag{3.19}$$

which ends the proof of Theorem 3.5, modulo proving Proposition 3.6. □

3.2.2 Proof of Proposition 3.6

We want to show that for any $\delta > 0$, one can take $\epsilon > 0$ sufficiently small so that for any $0 < a < \epsilon < \rho$,

$$\mathbb{E}\big[\big(\sum X_i^\epsilon - c\beta(\epsilon) \sum Y_i^\epsilon\big)^2\big] \le \delta\,.$$

Let us decompose this quantity as follows.

$$\begin{aligned}\mathbb{E}\big[\big(\sum X_i^\epsilon - c\beta(\epsilon)\sum Y_i^\epsilon\big)^2\big] &= \sum_{i,j}\mathbb{E}\big[\big(X_i^\epsilon - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j^\epsilon - c\beta(\epsilon)Y_j^\epsilon\big)\big] \\ &\le \sum_{i,j:d(B_i,B_j)\le r}\Big(\mathbb{E}\big[X_i^\epsilon X_j^\epsilon\big] + c^2\beta(\epsilon)^2\,\mathbb{E}\big[Y_i^\epsilon Y_j^\epsilon\big]\Big) \\ &+ \sum_{i,j:d(B_i,B_j)>r}\mathbb{E}\big[\big(X_i^\epsilon - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j^\epsilon - c\beta(\epsilon)Y_j^\epsilon\big)\big]\,,\end{aligned} \tag{3.20}$$

where r is a mesoscopic scale $\epsilon \ll r \ll \rho$ which will be chosen later. To go from the first to the second line, we used that fact that the cross product terms are necessarily negative as can be seen by first conditioning on the non-colored FK configuration ω_a.

The first term of the RHS of the above displayed inequality is easy to bound. Indeed,

$$\sum_{i,j:d(B_i,B_j)\le r}\mathbb{E}\big[X_i^\epsilon X_j^\epsilon\big] \le \sum_{x,y\in a\mathbb{Z}^2\cap Q \text{ s.t } d(x,y)\le 2r} a^{15/4}\mathbb{E}\big[\sigma_x\sigma_y\big] \le O(r^{7/4})\,,$$

and similarly for $\sum_{i,j:d(B_i,B_j)\le r} c^2\beta(\epsilon)^2\,\mathbb{E}\big[Y_i^\epsilon Y_j^\epsilon\big]$. One can thus fix $r>0$ small enough so that, uniformly in $a<\epsilon<r$, the first term in the RHS of (3.20) is $<\delta/2$.

For the second term, we proceed as in [GPS10] using a coupling argument. Proposition 3.6 will follow from the next lemma.

Lemma 3.7. *For any fixed $r<\rho<1$ and any $\tilde\delta>0$, one can choose $\epsilon=\epsilon(r,\rho,\tilde\delta)>0$ small enough such that for any pair of squares B_i,B_j with $d(B_i,B_j)>r$, one has*

$$\mathbb{E}\big[\big(X_i^\epsilon - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j^\epsilon - c\beta(\epsilon)Y_j^\epsilon\big)\big] \le \frac{\tilde\delta}{2}\mathbb{E}\big[X_i^\epsilon X_j^\epsilon\big]\,.$$

Let us explain why this lemma is enough to conclude the proof. Summing the estimate provided by the lemma over all B_i,B_j with $d(B_i,B_j)>r$, one gets

$$\sum_{i,j:d(B_i,B_j)>r}\mathbb{E}\big[\big(X_i^\epsilon - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j^\epsilon - c\beta(\epsilon)Y_j^\epsilon\big)\big] \le \frac{\tilde\delta}{2}\mathbb{E}\big[(L_Q^a(\bar\omega_a))^2\big]\,.$$

Now, it is straightforward to check that the second moment $\mathbb{E}\big[(L_Q^a(\bar\omega_a))^2\big]$ is bounded by $C\rho^{15/4}$ uniformly in $a<\epsilon<\rho$ where C is some universal constant. By choosing $\tilde\delta=\delta/C$, we conclude the proof. □

Proof of Lemma 3.7. Let us fix two squares B_i and B_j at distance at least r from each other. Conditioned on the event that both B_i and B_j are connected to $\partial(3Q)$, our strategy is to compare how things look within the ϵ-square B_i with the following "test case." Consider the ϵ-square B_0 centered at the origin and let Q_0 be the square $[-1,1]^2$ also centered at the origin. Let us define

$$X_0 := a^{15/8}\sum_{x\in a\mathbb{Z}^2\cap B_0}\sigma_x 1_{\{x\leftrightarrow\partial Q_0 \text{ in } \bar\omega_a\}}\,. \tag{3.21}$$

Recall the events $\mathcal{A}_1^{\pm}$ defined in Subsection 3.1.3 and applied here to the annulus $A = Q_0 \setminus B_0$. We first wish to show that there is a constant $c > 0$ such that

$$\begin{cases} \mathbb{E}\big[X_0 \mid A_1^+\big] \sim c\,\beta(\epsilon) \\ \mathbb{E}\big[X_0 \mid A_1^-\big] \sim -c\,\beta(\epsilon) \end{cases}, \tag{3.22}$$

uniformly as $0 < a < \epsilon$ go to 0. To see why this holds we note that, as in Section 4.5 in [GPS10], one has that

$$\mathbb{E}\big[X_0 \mid A_1^+\big] \sim (\epsilon/a)^2\, a^{15/8}\, \frac{\alpha_1^{\mathrm{FK}}(a,1)}{\alpha_1^{\mathrm{FK}}(\epsilon,1)}\,.$$

To adapt the proof from [GPS10], it is enough to have some control on the half-plane exponents of critical FK percolation "in the bulk,"which are obtained in the usual manner using RSW from [DHN11].

Now, using Theorem 1.3 (with $k = 0$) in [CHI12] together with Wu's result, Theorem 1.6, we have that $\alpha_1^{\mathrm{FK}}(a,1) \sim c\, a^{1/8}$ as $a \to 0$, which explains the desired asymptotic.

In what follows, for any $u \geq \epsilon$, we will denote by D_u ($\tilde{D}_u$) the square centered around B_i (B_j) of side-length u. Let us fix yet another mesoscopic scale γ so that $\epsilon \ll \gamma \ll r$ (for example $\gamma := r^2$). Let $m := d(B_i, B_j)/2$ and let z be the midpoint between the centers of B_i and B_j. Let R^+ be the event that $\{\partial D_\gamma \overset{\omega^+}{\longleftrightarrow} \partial D_m\} \cap \{$ there is a circuit of ω^+ inside $D_r \setminus D_\gamma$ that surrounds $D_\gamma\}$. The event R^- is defined similarly. Notice that $R^+ \cap R^- = \emptyset$. On the event $R^{\pm}$, let $C = C(\bar{\omega})$ be the outermost such open circuit for the FK configuration $\omega \in \mathscr{H}$ (the outermost open circuit necessarily has the appropriate color).

Let us analyse in the term $\mathbb{E}\big[\big(X_i - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j - c\beta(\epsilon)Y_j^\epsilon\big)\big]$ the contribution coming from the event $(R^+ \cup R^-)^c$, namely:

$$\mathbb{E}\big[\big(X_i - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j - c\beta(\epsilon)Y_j^\epsilon\big); (R^+ \cup R^-)^c\big] \leq \mathbb{E}\big[X_iX_j + c^2\beta(\epsilon)^2 Y_iY_j; (R^+ \cup R^-)^c\big]\,.$$

See the explanation after (3.20) as to why the cross product terms are negative. Following [GPS10],

$$\begin{aligned} &\mathbb{E}\big[X_iX_j; (R^+ \cup R^-)^c\big] \\ &= \mathbb{E}\big[X_iX_j;\, \partial D_\epsilon \overset{\omega}{\longleftrightarrow} \partial D_m;\, \partial \tilde{D}_\epsilon \overset{\omega}{\longleftrightarrow} \partial \tilde{D}_m;\, \partial B(z,2m) \overset{\omega}{\longleftrightarrow} \partial(3Q);\, (R^+ \cup R^-)^c\big] \\ &\leq O(1)\mathbb{E}^{\,\mathrm{wired}}\big[\tilde{X}_i\big]\mathbb{E}^{\,\mathrm{wired}}\big[\tilde{X}_j\big]\alpha_1^{\mathrm{wired}}(\epsilon,\rho)\alpha_1^{\mathrm{wired}}(\epsilon,\gamma)\mathbb{P}\big[\partial D_\gamma \overset{\omega}{\longleftrightarrow} \partial D_m;\, (R^+ \cup R^-)^c\big]\,, \end{aligned}$$

where we have just used FKG and where we dominated X_i by $\tilde{X}_i$, the number of points in B_i connected to ∂B_i (we also used some straightforward quasi-multiplicativity for the one-arm FK event which follows easily from the RSW theorem in [DHN11]; see for example [W09] for an explanation of quasi-multiplicativity in the case of standard percolation and see [CDH] for quasi-multiplicativity results in the case of FK percolation). Now, using FKG with RSW from [DHN11], we get that there exists an exponent $\xi > 0$ such that $\mathbb{P}\big[R^+ \cup R^- \mid \partial D_\gamma \overset{\omega}{\longleftrightarrow} \partial D_m\big] \geq 1 - (\gamma/m)^\xi$, which implies that

$$\mathbb{P}\big[\partial D_\gamma \overset{\omega}{\longleftrightarrow} \partial D_m;\, (R^+ \cup R^-)^c\big] \leq \alpha_1^{\mathrm{FK}}(\gamma,m)(\gamma/m)^\xi\,.$$

Altogether, we obtain that

$$\mathbb{E}\big[X_iX_j; (R^+ \cup R^-)^c\big] \leq O(1)\mathbb{E}\big[X_iX_j\big](\gamma/m)^\xi\,.$$

The term $\mathbb{E}\big[c^2\beta(\epsilon)^2 Y_i Y_j; (R^+\cup R^-)^c\big]$ can be treated similarly. We may thus focus our analysis on what is happening on the event $R^+\cup R^-$. Let $\mathcal{F}_C$ be the filtration induced by the configuration outside the contour C. One can write

$$
\begin{aligned}
&|\mathbb{E}\big[\big(X_i - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j - c\beta(\epsilon)Y_j^\epsilon\big); R^+\cup R^-\big]| \\
&\qquad\qquad = \mathbb{E}\big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C\big]; R^+\cup R^-\big]\,,
\end{aligned}
$$

since on the event $R^+\cup R^-$, the variable $|X_j - c\beta(\epsilon)Y_j|$ is measurable w.r.t. $\mathcal{F}_C$. Now,

$$
\begin{aligned}
&\mathbb{E}\big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C\big]; R^+\cup R^-\big] \\
&\qquad = \mathbb{P}\big[R^+\big]\mathbb{E}\big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C\big] \bigm| R^+\big] \\
&\qquad + \mathbb{P}\big[R^-\big]\mathbb{E}\big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C\big] \bigm| R^-\big]
\end{aligned}
$$

Let us analyze the first term, it gives

$$
\begin{aligned}
&\mathbb{P}\big[R^+\big]\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C, R^+\big] \Bigm| R^+\Big] \\
&\quad = \mathbb{P}\big[R^+\big]\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{1}_{C\leftrightarrow\partial(3Q)}\mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| C, R^+\big] \Bigm| R^+\Big] \\
&\quad = \mathbb{P}\big[R^+\big]\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{1}_{C\leftrightarrow\partial(3Q)}\mathbb{P}\big[\partial(B_i)\leftrightarrow C \bigm| C, R^+\big]\mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| C, \partial(B_i)\leftrightarrow C\big] \Bigm| R^+\Big]\,.
\end{aligned}
$$

We will prove below the following lemma.

Lemma 3.8 (Coupling lemma)**.** *For any contour C, we have the following control on the conditional expectation:*

$$
\mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| C, \partial(B_i)\leftrightarrow C\big] \le K(\epsilon/\gamma)^\alpha\beta(\epsilon)
$$

for some exponent $\alpha > 0$ and some constant $K\in(0,\infty)$.

Plugging this lemma into the last displayed equation leads to

$$
\begin{aligned}
&\mathbb{P}\big[R^+\big]\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{E}\big[|X_i - c\beta(\epsilon)Y_i| \bigm| \mathcal{F}_C, R^+\big] \Bigm| R^+\Big] \\
&\quad \le C(\epsilon/\gamma)^\alpha\beta(\epsilon)\mathbb{P}\big[R^+\big]\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{1}_{C\leftrightarrow\partial(3Q)}\mathbb{P}\big[\partial(B_i)\leftrightarrow C \bigm| C, R^+\big] \Bigm| R^+\Big] \\
&\quad \le O(1)(\epsilon/\gamma)^\alpha\beta(\epsilon)\alpha_1^{\mathrm{FK}}(\gamma,m)\,\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{1}_{C\leftrightarrow\partial(3Q)}\mathbb{P}\big[\partial(B_i)\leftrightarrow C \bigm| C, R^+\big] \Bigm| R^+\Big]\,. \qquad (3.23)
\end{aligned}
$$

Now, similarly to the above analysis of what happens on the event $(R^+\cup R^-)^c$, it is not hard to check by cutting into different scales and dominating by wired boundary conditions that

$$
\mathbb{E}\Big[|X_j - c\beta(\epsilon)Y_j|\, \mathbb{1}_{C\leftrightarrow\partial(3Q)}\mathbb{P}\big[\partial(B_i)\leftrightarrow C \bigm| C, R^+\big] \Bigm| R^+\Big] \le O(1)\beta(\epsilon)\alpha_1^{\mathrm{FK}}(\epsilon,\rho)\,\alpha_1^{\mathrm{FK}}(\epsilon,\gamma)\,,
$$

which together with (3.23) and quasi-multiplicativity gives us (since one has also the same estimate on the event R^-),

$$
\begin{aligned}
|\mathbb{E}\big[\big(X_i - c\beta(\epsilon)Y_i^\epsilon\big)\big(X_j - c\beta(\epsilon)Y_j^\epsilon\big); R^+\cup R^-\big]| &\le O(1)(\epsilon/\gamma)^\alpha\beta(\epsilon)^2\frac{\alpha_1^{\mathrm{FK}}(\epsilon,\rho)^2}{\alpha_1^{\mathrm{FK}}(m,\rho)} \\
&\le O(1)(\epsilon/\gamma)^\alpha\mathbb{E}\big[X_iX_j\big]\,,
\end{aligned}
$$

which (modulo proving Lemma 3.8) completes our proof of Lemma 3.7. □

3.2.3 Proof of Lemma 3.8

Let ν_C be the wired FK probability measure conditioned on $\partial B_i \leftrightarrow C$ and let ν_0 be the FK probability measure in Q_0 conditioned on the event $\mathcal{A}_1 = \mathcal{A}_1(\tilde{x} + Q_0 \setminus B_0)$, where we translated the annulus $A = Q_0 \setminus B_0$ so that it surrounds B_i. Clearly, in the domain $\mathcal{D}_C$ (inside the circuit C), the measure ν_C dominates ν_0. Using RSW from [DHN11], there is an open circuit in $\mathcal{D}_C \setminus B_i$ for $\omega_a^0 \sim \nu_0$ with ν_0-probability at least $1 - c(\epsilon/\gamma)^\xi$. Let us call this event W. On the event W, let $\tilde{C}$ be the outermost circuit inside $\mathcal{D}_C$ for ω_a^0. Since ν_C dominates ν_0, one can couple $\omega_a^C \sim \nu_C$ with ω_a^0 so that on the event W, they share the same open circuit $\tilde{C}$ and are conditioned inside $\mathcal{D}_{\tilde{C}}$ only on the constraint $\{\partial B_i \leftrightarrow \tilde{C}\}$; in particular, on the event W, in this coupling one has $X_i = X_0$. In order to prove Lemma 3.8, it is enough to show that $\mathbb{E}\big[X_i; W^c \bigm| C, \partial B_i \leftrightarrow C\big]$ and $\mathbb{E}\big[X_0; W^c \bigm| C, \partial B_i \leftrightarrow C\big]$ are negligible w.r.t $\beta(\epsilon)$, which is straightforward using the quantity $\mathbb{E}\big[\tilde{X}_i\big] \times \mathbb{P}\big[W^c\big]$ as we did previously while analyzing what happened on the event $(R^+ \cup R^-)^c$. □

3.3 Approximating Φ^∞ using signed measures

There are several advantages to the above proof. First, it shows that Φ^∞ is measurable w.r.t. $\bar{\omega}_\infty$. Furthermore, it gives us a good way to visualize the limiting magnetization field Φ^∞. For example the following result can be extracted from our proof above.

Theorem 3.9 (under Assumption 3.2)**.** *Let $\Phi^\infty = \Phi^\infty(\bar{\omega}_\infty)$ be the scaling limit of the magnetization field in the domain $[0,1]^2$. For each cut-off $\rho > 0$, let $\mathcal{C} = \mathcal{C}_\rho(\bar{\omega}_\infty)$ be the a.s. finite set of clusters of $\bar{\omega}_\infty$ which cross at least one annulus $3Q \setminus Q$ with Q a square in S_ρ.*

For each cluster $C \in \mathcal{C}$, using the technology of our above proof (see also [GPS10]), one can define an **area measure** *μ_C supported on this cluster which is measurable w.r.t. $\bar{\omega}_\infty$. Furthermore this ensemble of area measures is such that the signed measure $\Phi^\rho = \Phi^\rho(\bar{\omega}_\infty)$ defined as follows*

$$\Phi^\rho(\bar{\omega}_\infty) := \sum_{S \in \mathcal{C}_\rho(\bar{\omega}_\infty)} \sigma_C \, \mu_C \,, \tag{3.24}$$

approximates our magnetization field Φ^∞ in the following quantitative manner:

$$\mathbb{E}\big[\|\Phi^\infty - \Phi^\rho\|_{\mathcal{H}^{-3}}^2\big] \le O(\rho^{7/4})\,.$$

Very briefly, the latter quantitative statement follows from a computation similar to what was done in Subsection 2.1. Indeed, one has

$$\begin{aligned}
\mathbb{E}\big[\|\Phi^\infty - \Phi^\rho\|_{\mathcal{H}^{-3}}^2\big] &= \limsup_{a \to 0} \mathbb{E}\big[\|\Phi(\bar{\omega}_a) - \Phi^\rho(\bar{\omega}_a)\|_{\mathcal{H}^{-3}}^2\big] \\
&= \limsup_{a \to 0} \sum_{j,k \ge 1} \frac{\mathbb{E}\big[\langle(\Phi(\bar{\omega}_a) - \Phi^\rho(\bar{\omega}_a)), e_{j,k}\rangle^2\big]}{(j^2 + k^2)^3}\,.
\end{aligned}$$

Further, for each fixed $j, k \geq 1$,

$$\begin{aligned}
\mathbb{E}\big[\langle(\Phi(\bar\omega_a) - \Phi^\rho(\bar\omega_a)), e_{j,k}\rangle^2\big] \leq a^{15/4} \sum_{x,y} \Big| \iint_{S_a(x)\times S_a(y)} & \frac{\mathbb{E}\big[\sigma_x\,\sigma_y 1_{x\notin\mathcal{C}, y\notin\mathcal{C}}\big]}{a^4} e_{j,k}(\bar x) e_{j,k}(\bar y) dA(\bar x) dA(\bar y)\Big| \\
+ a^{15/4} \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} & \Big(\int_{S_a(x)} \frac{\mathbb{P}\big[x\notin\mathcal{C}\big]}{a^2} e_{j,k}(x) dA(\bar x)\Big)^2 . \\
\leq & a^{15/4}\|e_{j,k}\|_\infty^2 \sum_{x\neq y, |x-y|\leq\rho, \in[0,1]^2\cap a\mathbb{Z}^2} |\mathbb{E}\big[\sigma_x\,\sigma_y\big]| + a^{15/4}\|e_{j,k}\|_\infty^2 \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} 1 \\
\leq & O(\rho^{7/4}) ,
\end{aligned}$$

for $0 < a < \rho$ as $\rho \to 0$.

This theorem gives an idea of how regular or irregular the random distribution Φ^∞ should typically be. For instance, this theorem should be useful if we wanted to "detect" which Sobolev space $\mathcal{H}^{-\epsilon}$, Φ^∞ should a.s. be in; see Remark 2.5.

4 Second proof of the scaling limit of Φ^a using the n-point functions of Chelkak, Hongler and Izyurov

In this part, we will give a different proof of Theorem 1.2, using the recent breakthrough results of Chelkak, Hongler and Izyurov in [CHI12]. From our tightness result obtained in Section 2, recall that there exist subsequential scaling limits $\Phi^\star = \lim \Phi^{a_n}$ for the convergence in law in the space $\mathcal{H}^{-3}$. We wish to prove that there is a unique such subsequential scaling limit. For this, we will use the following classical fact (see for example [LT11]).

Proposition 4.1. *If h is a random distribution in $\mathcal{H}^{-3}$ (for the sigma-field generated by the topology of $\|\cdot\|_{\mathcal{H}^{-3}}$), then the law of h is uniquely characterized by*

$$\phi_h(f) := \mathbb{E}\big[e^{i\,\langle h,f\rangle}\big] ,$$

as a function of $f \in \mathcal{H}^3$.

Using the tightness property proved in Section 2, Theorem 1.2 will thus follow from the next result.

Proposition 4.2. *For any $f \in \mathcal{H}^3$, the quantity*

$$\phi_{\Phi^a}(f) = \mathbb{E}\big[e^{i\,\langle\Phi^a,f\rangle}\big]$$

converges as the mesh size $a \searrow 0$.

The proof of this proposition will be divided into two main steps as follows

1. First, we will show that Φ^a has "uniform exponential moments" which will allow us to express its characteristic function using

$$\phi_{\Phi^a}(f) = \mathbb{E}\big[e^{i\,\langle\Phi^a,f\rangle}\big] = 1 + \sum_{k\geq 1} \frac{i^k\,\mathbb{E}\big[\langle\Phi^a, f\rangle^k\big]}{k!} .$$

2. Then, it remains to compute each k^{th} moment $\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]$, i.e., to show uniqueness as $a \to 0$. For this, one uses the scaling limit results from [CHI12] together with Proposition 4.9 below which takes care of k-tuples of points in the plane where at least two points are close to each other.

Let us now state the main result we will use from [CHI12].

Theorem 4.3 ([CHI12], Theorem 1.3). *Let Ω be a bounded simply connected domain, and Ω_a be discretizations of Ω (built from $\Omega \cap a\,\mathbb{Z}^2$). We denote by ξ the boundary conditions chosen on Ω, and we assume ξ to be either $+$ or* free *here. Then, for any $k \geq 1$, there exist k-***point functions**

$$z_1, \ldots, z_k \in \Omega^k \mapsto \langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^\xi \,,$$

so that for any $\epsilon > 0$, as the mesh size $a \to 0$ and uniformly over all $z_1, \ldots, z_k \in \Omega$ at distance at least ϵ from $\partial\Omega$ and from each other, one has

$$\varrho(a)^{-k/2} \cdot \mathbb{E}_{\Omega_a}^\xi\big[\sigma_{z_1} \ldots \sigma_{z_k}\big] \to \langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^\xi \,. \tag{4.1}$$

(Recall that $\varrho(a)$ is the renormalization factor defined in (1.1).)

Furthermore, the functions $\langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^\xi$ are conformally covariant in the following sense: if $\phi : \Omega \to \Omega'$ is a conformal map, then

$$\langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^\xi = \langle \sigma_{\phi(z_1)} \ldots \sigma_{\phi(z_k)} \rangle_{\Omega'}^\xi \prod |\phi'(z_i)|^{1/8} \,.$$

Remark 4.4.

- It is noted in [CHI12] that although their Theorem 1.3 is stated only for plus boundary conditions, the conclusions are valid for free and other boundary conditions as well.
- In [CHI12], the discretization is slightly different, which results in changing here the k-point function $\langle \sigma_{z_1} \ldots , \sigma_{z_k} \rangle_\Omega^\xi$ by a constant factor.
- In most of this paper, we assume Wu's result, Theorem 1.6. In particular, one can then use the above theorem with $a^{k/8}$ instead of $\varrho(a)^{k/2}$ (and with yet a further change of the k point function by another scalar). See Section 6 for the analysis when one does not wish to assume Wu's result.

4.1 Exponential moments for the magnetization random variable

In this section, we shall show that if m^a denotes the magnetization random variable $\langle \Phi^a, 1_{[0,1]^2} \rangle$ (for wired or free boundary conditions on the square $[0,1]^2$), then m^a has exponential moments. More precisely, we will prove:

Proposition 4.5. *For any $t \in \mathbb{R}$, and for any boundary condition ξ on $[0,1]^2$, one has*

$$\limsup_{a \searrow 0} \mathbb{E}^{\,\xi}\big[e^{t\, m^a}\big] < \infty \,.$$

There are a number of ways to prove this proposition. We present one based on the Griffiths-Hurst-Sherman inequality from [GrHS70]. Let us state it here.

Theorem 4.6 (GHS inequality, [GrHS70]). *Let $G = (V, E)$ be a finite graph. Consider a pair ferromagnetic Ising model on this graph (i.e., the interactions J_{ij} between vertices $i \sim j$ are non-negative) and assume furthermore that the external field $\mathbf{h} = (h_v)_{v \in V}$ (which may vary from one vertex to another) is also non-negative. Under these general assumptions, one has for any vertices $i, j, k \in V$,*

$$\langle \sigma_i \sigma_j \sigma_k \rangle - \Big(\langle \sigma_i \rangle \langle \sigma_j \sigma_k \rangle + \langle \sigma_j \rangle \langle \sigma_i \sigma_k \rangle + \langle \sigma_k \rangle \langle \sigma_i \sigma_j \rangle \Big) + 2 \langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle \le 0 \,.$$

This inequality has the following useful corollary (see, e.g., [CGN12]).

Corollary 4.7. *Let $G = (V, E)$ be a finite graph and let $K \subset V$ be a non-empty subset of the vertices. Let us consider a ferromagnetic Ising model on G with the spins in K prescribed to be $+$ spins and with a constant magnetic field $h \ge 0$ on $V \setminus K$. Then the partition function of this model, i.e.,*

$$Z_{\beta,h} := \sum_{\sigma \in \{-,+\}^{V \setminus K}} \exp \left(-\beta E(\sigma) + h \sum_{i \in V \setminus K} \sigma_i \right) ,$$

where $E(\sigma) = \sum_{i \sim j \in V} J_{ij} \sigma_i \sigma_j$, satisfies

$$\partial_h^3 \log(Z_{\beta,h}) \le 0 \,.$$

Proof of Proposition 4.5. If $t < 0$, using the symmetry of the Ising model, by changing the boundary condition ξ into $-\xi$, we can assume $t > 0$. Hence one may assume that $t \ge 0$. This makes the function $x \mapsto e^{t\,x}$ increasing, and one can thus use the FKG inequality which implies that for any $t \ge 0$ and any boundary condition ξ, one has

$$\mathbb{E}^{\,\xi} \big[e^{t\, m^a} \big] \le \mathbb{E}^{\,+} \big[e^{t\, m^a} \big] \,.$$

With $+$ boundary condition on $[0,1]^2 \cap a\mathbb{Z}^2$, one can now rely on the above corollary of the GHS inequality which yields

$$\begin{aligned}
\partial_h^3 \Big[\log(\sum e^{\beta E(\sigma) + h \sum \sigma_i}) \Big] &= \partial_h^3 \Big[\log(\frac{\sum e^{\beta E(\sigma) + h \sum \sigma_i}}{\sum e^{\beta E(\sigma)}}) \Big] \\
&= \partial_h^3 \Big[\log \mathbb{E}_\beta \big[e^{h \sum \sigma_i} \big] \Big] \\
&\le 0 \,.
\end{aligned}$$

With $\beta = \beta_c$ and $h := t\, a^{15/8}$, one obtains that for any $t \ge 0$ and any mesh size $a > 0$:

$$\partial_t^3 \log \mathbb{E}^{\,+} \big[e^{t\, m^a} \big] \le 0 \,. \tag{4.2}$$

Now let $\phi(t) := \mathbb{E}^{\,+} \big[e^{t\, m^a} \big]$. It is easy to check that

$$\begin{cases} \phi'(0) = \mathbb{E}^{\,+} \big[m^a \big] \\ \phi''(0) = \mathbb{E}^{\,+} \big[(m^a - \langle m^a \rangle)^2 \big] \,. \end{cases}$$

This, together with (4.2), implies that for any $t \geq 0, a > 0$:

$$\log \mathbb{E}^+\big[e^{t\, m^a}\big] \leq t\, \mathbb{E}^+\big[m^a\big] + \frac{t^2}{2}\mathbb{E}^+\big[(m^a - \langle m^a \rangle)^2\big]\,.$$

By our choice of rescaling, $m^a := a^{15/8} \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} \sigma_x$, we know from Proposition A.1 in the appendix that $\sup_{a>0} t\, \mathbb{E}^+\big[m^a\big] + \frac{t^2}{2}\mathbb{E}^+\big[(m^a - \langle m^a\rangle)^2\big] = O(t+t^2) < \infty$, which completes the proof of Proposition 4.5. $\square$

We have the following easy corollary of Proposition 4.5; it applies for example to $\Omega = [0,1]^2$ with ξ plus or free, where there is a unique limit, and also to quite general Ω and ξ where there may only be limits along subsequences of $a \to 0$.

Corollary 4.8. *If $m = m^\xi$ is the limit in law of m^a for some Ω and ξ, then*

(i) $\mathbb{E}\big[e^{t\, m}\big] < \infty$;

(ii) furthermore, as $a \to 0$, $\mathbb{E}\big[e^{t\, m^a}\big] \to \mathbb{E}\big[e^{t\, m}\big]$.

The proof is straightforward. Note that for any $t \in \mathbb{R}$, by Fatou's lemma one has that

$$\mathbb{E}\big[e^{t\, m}\big] \leq \liminf_{a\to 0} \mathbb{E}\big[e^{t\, m^a}\big]\,, \tag{4.3}$$

which implies (i). Now (ii) follows easily from (i) (used with some $\tilde t > |t|$ and with $\xi = +$), FKG, and the weak convergence of m^a to m.

4.2 Computing the characteristic function

Let us prove Proposition 4.2 assuming Proposition 4.9 below. Let $f \in \mathcal{H}^3$ be fixed once and for all. For any $k \geq 1$, note that

$$\begin{aligned}
|\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]| &= |\mathbb{E}\big[\big(\sum_{x\in[0,1]^2\cap a\,\mathbb{Z}^2} a^{15/8} f(x)\sigma_x\big)^k\big]| \\
&= |\, a^{15k/8} \sum_{z_1,\ldots,z_k \in ([0,1]^2\cap a\mathbb{Z}^2)^k} f(z_1)\ldots f(z_k)\, \mathbb{E}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big]| \\
&\leq \|f\|_\infty^k\, a^{15k/8} \sum_{z_1,\ldots,z_k \in ([0,1]^2\cap a\mathbb{Z}^2)^k} \mathbb{E}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big] \\
&\leq \|f\|_\infty^k\, \mathbb{E}\big[(m^a)^k\big]\,.
\end{aligned}$$

Since, by Proposition 4.5, m^a has uniform exponential moments, we deduce that the series

$$\phi_{\Phi^a}(f) = \mathbb{E}\big[e^{i\,\langle \Phi^a, f\rangle}\big] = 1 + \sum_{k\geq 1} \frac{i^k\, \mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]}{k!}$$

is indeed summable. Now, for each $k \geq 1$, let us prove that the k-th moment $\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]$ has a limit as $a \to 0$. Let us fix some cut-off $\epsilon > 0$ and let us divide the k-th moment as follows:

$$\begin{aligned}\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big] &= \sum_{z_1,\ldots,z_k \in ([0,1]^2 \cap a\mathbb{Z}^2)^k} a^{15k/8}\, f(z_1)\ldots f(z_k)\, \mathbb{E}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big] \\ &= \sum_{\substack{z_1,\ldots,z_k \\ |z_i - z_j| \geq \epsilon\ \forall i \neq j}} a^{2k} f(z_1)\ldots f(z_k)\, a^{-k/8}\mathbb{E}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big] \\ &+ \sum_{\substack{z_1,\ldots,z_k \\ \inf_{i\neq j}|z_i - z_j| < \epsilon}} a^{15k/8} f(z_1)\ldots f(z_k)\, \mathbb{E}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big]\,. \end{aligned} \tag{4.4}$$

Using Theorem 4.3 and assuming Wu's result, we have that in the domain $[0,1]^2$, there exists a function $z_1,\ldots,z_k \mapsto \langle z_1,\ldots,z_k\rangle_{[0,1]^2}$ such that

$$a^{-k/8}\, \mathbb{E}_{[0,1]^2_a}\big[\sigma_{z_1}\ldots\sigma_{z_k}\big] \to \langle z_1,\ldots,z_k\rangle_{[0,1]^2}\,, \tag{4.5}$$

uniformly in $\inf_{i\neq j}|z_i - z_j| \geq \epsilon$ (again, up to a change by a deterministic scalar in the definition of these functions which arises from normalizing by either $\varrho(a)^{k/2}$ or $a^{k/8}$). The fact that the convergence is uniform implies that the first term in equation (4.4) converges as the mesh size $a \to 0$ to

$$\iint_{\substack{z_1,\ldots,z_k \in ([0,1]^2)^k \\ |z_i - z_j| \geq \epsilon\ \forall i\neq j}} f(z_1)\ldots f(z_k)\langle z_1,\ldots,z_k\rangle_{[0,1]^2}\, dz_1\ldots dz_k\,.$$

To conclude the proof, it remains to prove that the second term in Equation (4.4) is small uniformly in $0 < a < \epsilon$, when the cut-off ϵ is small. This is the content of the next section.

4.3 Handling the "local" k-tuples

Proposition 4.9. *Let Ω be a domain with $+$ boundary conditions. For any $k \geq 1$, there exist constants $C_k = C_k(\Omega) < \infty$ such that, for all $0 < a < \epsilon$,*

$$\sum_{(x_1,\ldots,x_k):\inf_{i\neq j}\{|x_i - x_j|\} < \epsilon} a^{15k/8}\, \mathbb{E}\Big[\prod_1^k \sigma_{x_i}\Big] \leq C_k\, \epsilon^{7/4}\,.$$

Proof. Our proof is based on the FK representation; we remark that a somewhat different proof can be obtained by using the Gaussian correlation inequalities of [N75]. One implements the $+$ boundary condition via a ghost vertex corresponding to the boundary and then reduces estimates of kth moments essentially to one and two point correlations. Those are handled by arguments like in the appendix below — see especially Equation A.3. We now proceed with more details using the FK representation approach.

One can write $\mathbb{E}\big[\prod_1^k \sigma_{x_i}\big]$ using FK as follows: let Δ_k be the set of graphs Γ defined on the set of vertices $V_k := \{1,\ldots,k\} \cup \{+\}$, and which are such that the clusters of Γ which do not contain the point $+$ are all of even size. (Of course the number $|\Delta_k|$ of such graph structures is finite).

Now, similarly to Wick's theorem, one has the identity

$$\mathbb{E}\big[\prod_1^k \sigma_{x_i}\big] = \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[A_{x_1,\ldots,x_k}(\Gamma)\big]\,,$$

where $A_{x_1,\ldots,x_k}(\Gamma)$ is the event that the graph structure induced by the FK configuration ω on the set $\{x_1,\ldots,x_k\}\cup\{\partial\Omega\}$ is given by the graph $\Gamma\in\Delta_k$.

Note that if Γ is not connected, there is some negative information inherent to the event $A_{x_1,\ldots,x_k}(\Gamma)$. To overcome this, let $\bar{A}_{x_1,\ldots,x_k}(\Gamma)$ be the event that the graph induced by the FK configuration ω on $\{x_1,\ldots,x_k\}\cup\{\partial\Omega\}$ *includes* the graph Γ. Defined this way, $\bar{A}_{x_1,\ldots,x_k}(\Gamma)$ is an increasing event (which will allow us to use FKG) and one has for any $\Gamma\in\Delta_k$:

$$\mathbb{P}\big[A_{x_1,\ldots,x_k}(\Gamma)\big] \le \mathbb{P}\big[\bar{A}_{x_1,\ldots,x_k}(\Gamma)\big]\,.$$

Therefore, it is enough for us to prove the following upper bound:

$$\sum_{(x_1,\ldots,x_k):\inf_{i\neq j}\{|x_i-x_j|\}<\epsilon}\ \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[\bar{A}_{x_1,\ldots,x_k}(\Gamma)\big]\ \le C\epsilon^{7/4}a^{-\frac{15k}{8}}\,.$$

This is the subject of the next lemma. □

Lemma 4.10. *For any domain Ω and any $k\ge 1$, there exists a constant $C_k = C_k(\Omega)<\infty$ such that, for all $0<a<\epsilon$, one has:*

(i)

$$\sum_{x_1,\ldots,x_k\in\Omega_a}\ \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[\bar{A}_{x_1,\ldots,x_k}(\Gamma)\big] \le C_k\, a^{-\frac{15k}{8}}\,,$$

(ii)

$$\sum_{(x_1,\ldots,x_k):\inf_{i\neq j}\{|x_i-x_j|\}<\epsilon}\ \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[\bar{A}_{x_1,\ldots,x_k}(\Gamma)\big]\ \le C_k\epsilon^{7/4}a^{-\frac{15k}{8}}\,.$$

Proof (sketch). The proof of this lemma proceeds by induction. For $k=1$, the bounds follow easily from Proposition A.1 in the appendix. For $k=2$, using again Proposition A.1 and summing $\mathbb{P}\big[x_1\leftrightarrow x_2\big]$ over all x_1,x_2 which are such that $|x_1-x_2|\in(2^{-b-1},2^{-b}]$, one gets a bound of the form $O(1)a^{-4}2^{-2b}(2^{-b}/a)^{-1/4} = O(1)2^{-7b/4}a^{-15/4}$, where $a^{-4}2^{-2b} = a^{-2}(2^{-b}/a)^2$ comes from the number of ways one can choose x_1 and x_2. Summing over all possible values of b smaller than $\log_2(a^{-1})$ gives the first bound, while summing over values of b such that $\log_2(\epsilon^{-1})\le b\le\log_2(a^{-1})$ gives the second bound. (We neglect boundary issues that can easily be dealt with.)

Let now $k\ge 3$ and assume that property (i) holds for all $k'<k$. We will first prove that it implies property (ii) from which (i) easily follows (in fact formally (i) readily follows from (ii) by taking ϵ large enough but due to boundary issues, it is better to divide the study into these two sums).

The outer sum in (ii) is over the ordered k-tuples $(x_1,\ldots,x_k)$ which are such that $l:=\inf_{i\neq j}|x_i-x_j|<\epsilon$. For any such k-tuple $(x_1,\ldots,x_k)$, let us choose one point among all points which are at

distance $\inf_{i\neq j}|x_i - x_j|$ from at least one of the others (there are at most k ways to pick one) and let us reorder the points into a k-tuple $(\hat{x}_1, \ldots, \hat{x}_k)$ so that the point we have chosen is $\hat{x}_1$.

This way, we obtain

$$\sum_{\substack{x_1,\ldots,x_k\in\Omega_a\\ \inf_{i\neq j}|x_i-x_j|<\epsilon}} \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[\bar{A}_{x_1,\ldots,x_k}(\Gamma)\big]$$

$$\leq k \sum_{\substack{\hat{x}_1,\ldots,\hat{x}_k\\ \inf_{i\neq j}|\hat{x}_i-\hat{x}_j| = \inf_{i\neq 1}|\hat{x}_1-\hat{x}_i|<\epsilon}} \sum_{\Gamma\in\Delta_k} \mathbb{P}\big[\bar{A}_{\hat{x}_1,\ldots,\hat{x}_k}(\Gamma)\big] .$$

Now, for any such $(\hat{x}_1, \ldots, \hat{x}_k)$, we split the sum over $\Gamma \in \Delta_k$ in two parts.

(1) Consider first the sum over graphs Γ such that the cluster of $\hat{x}_1$ in Γ contains a point $\hat{x}_m$ at distance $< 2\epsilon$ from $\hat{x}_1$. Again by reordering (and possibly losing a factor of k), one can assume that $\hat{x}_m = \hat{x}_2$. Now let $A_{\hat{x}_1,\hat{x}_2}$ be an annulus which surrounds $\hat{x}_1$ and $\hat{x}_2$ and which is such that, by RSW, there is probability $c > 0$ of the event $S = S(A_{\hat{x}_1,\hat{x}_2})$ that there is an open path in $A_{\hat{x}_1,\hat{x}_2}$ surrounding $\hat{x}_1$ and $\hat{x}_2$.

Let $\hat{\Gamma}$ be a graph on $\{\hat{x}_3, \ldots, \hat{x}_k\}$ obtained from Γ in the following way. If the cluster of $\hat{x}_1$ and $\hat{x}_2$ in Γ does not contain other points, let $\hat{\Gamma} = \Gamma \setminus \{\hat{x}_1, \hat{x}_2\}$. Otherwise, first add some connection, if necessary, to make the cluster of $\hat{x}_1$ and $\hat{x}_2$ in Γ connected without using $\hat{x}_1$ and $\hat{x}_2$ (i.e., all other vertices are connected by paths that do not pass through $\hat{x}_1$ and $\hat{x}_2$), and then remove $\hat{x}_1$ and $\hat{x}_2$ from the cluster. Note that, in both cases, $\hat{\Gamma} \in \Delta_{k-2}$. Using FKG, one can easily check that

$$\begin{aligned}
\mathbb{P}\big[\bar{A}_{\hat{x}_1,\ldots,\hat{x}_k}(\Gamma)\big] &\leq (1/c)\,\mathbb{P}\big[\bar{A}_{\hat{x}_1,\ldots,\hat{x}_k}(\Gamma) \text{ and S}\big]\\
&\leq (1/c)\,\mathbb{P}\big[\hat{x}_1 \leftrightarrow \hat{x}_2 \text{ and } \bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma}) \text{ and S}\big]\\
&\leq (1/c)\,\mathbb{P}^+\big[\hat{x}_1 \leftrightarrow \hat{x}_2\big]\mathbb{P}\big[\bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma})\big]\\
&\leq O(1)d^{-1/4}a^{1/4}\,\mathbb{P}\big[\bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma})\big] ,
\end{aligned}$$

where d denotes the distance between $\hat{x}_1$ and $\hat{x}_2$ and by $+$ we mean wired b.c. on the inner boundary of $A_{\hat{x}_1,\hat{x}_2}$.

Summing over all $\hat{x}_1, \ldots, \hat{x}_k$ which are such that $d = |\hat{x}_1 - \hat{x}_2| \in (2^{-b-1}, 2^{-b}]$, and considering that there are at most k^2 ways of choosing $\hat{x}_1$ and $\hat{x}_2$ from $\{x_1, \ldots, x_k\}$, this case gives a contribution which is bounded by

$$O(1)k^2 a^{-4} 2^{-2b} 2^{b/4} a^{1/4} C_{k-2} a^{-\frac{15(k-2)}{8}} ,$$

where $a^{-2}(2^{-b}/a)^2 = a^{-4}2^{-2b}$ is an upper bound on the number of ways to choose $\hat{x}_1$ and $\hat{x}_2$ from Ω_a. Hence, we get the following upper bound:

$$O(1)k^2 2^{-7b/4} C_{k-2} a^{-\frac{15k}{8}} .$$

It remains to sum over the possible values of b, i.e., $\log_2(\epsilon^{-1}) \leq b \leq \log_2(a^{-1})$, which gives a bound of the desired form.

Note that we neglected boundary issues here (they can be handled easily at least if $\partial\Omega$ is smooth enough).

(2) Consider now the remaining sum over graphs Γ such that the cluster of $\hat{x}_1$ in Γ doesn't contain any point at distance $< 2\epsilon$ from $\hat{x}_1$. In this case there is at least one point, say $\hat{x}_2$, which is at distance l from $\hat{x}_1$. If the cluster of $\hat{x}_2$ in Γ contains a point at distance $< 2\epsilon$ from $\hat{x}_2$, then we can take $\hat{x}_2$ to play the role of $\hat{x}_1$ and we are back in situation 1. We can therefore assume that the cluster of $\hat{x}_2$ in Γ doesn't contain any point at distance $< 2\epsilon$ from $\hat{x}_2$. We can then pick an annulus $A_{\hat{x}_1,\hat{x}_2}$ that surrounds $\hat{x}_1$ and $\hat{x}_2$ and does not contain any other point belonging to the clusters of $\hat{x}_1$ and $\hat{x}_2$ in Γ, and which, by RSW, contains an open path surrounding $\hat{x}_1$ and $\hat{x}_2$ with probability $c > 0$. We call $S = S(A_{\hat{x}_1,\hat{x}_2})$ the latter event. If S occurs, $\hat{x}_1$ and $\hat{x}_2$ belong to the same FK cluster. If we denote by $\hat{\Gamma}$ a graph on $\{\hat{x}_3, \ldots, \hat{x}_k\}$ obtained from Γ by connecting the clusters of $\hat{x}_1$ and $\hat{x}_2$ in Γ outside of $\hat{x}_1$ and $\hat{x}_2$, and then removing $\hat{x}_1$ and $\hat{x}_2$ from Γ, we have that $\hat{\Gamma} \in \Delta_{k-2}$. Using FKG, one can easily check that

$$\begin{aligned}
\mathbb{P}\big[\bar{A}_{\hat{x}_1,\ldots,\hat{x}_k}(\Gamma)\big] &\leq (1/c)\,\mathbb{P}\big[\bar{A}_{\hat{x}_1,\ldots,\hat{x}_k}(\Gamma) \text{ and S}\big] \\
&\leq (1/c)\,\mathbb{P}\big[\hat{x}_1 \leftrightarrow \hat{x}_2 \text{ and } \bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma}) \text{ and S}\big] \\
&\leq (1/c)\,\mathbb{P}^+\big[\hat{x}_1 \leftrightarrow \hat{x}_2\big]\mathbb{P}\big[\bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma})\big] \\
&\leq O(1) l^{-1/4} a^{1/4}\,\mathbb{P}\big[\bar{A}_{\hat{x}_3,\ldots,\hat{x}_k}(\hat{\Gamma})\big]\,,
\end{aligned}$$

where by $+$ we mean wired on the inner boundary of $A_{\hat{x}_1,\hat{x}_2}$.

Summing over all $x_1, \ldots, x_k$ which are such that $l = |\hat{x}_1 - \hat{x}_2| = \inf_{i \neq j} |x_i - x_j| \in (2^{-b-1}, 2^{-b}]$, this case gives a contribution which is bounded by

$$O(1) k^2 a^{-4} 2^{-2b} 2^{b/4} a^{1/4} C_{k-2} a^{-\frac{15(k-2)}{8}}\,,$$

where k^2 comes from the ways of choosing $\hat{x}_1$ and $\hat{x}_2$ from $\{x_1, \ldots, x_k\}$ and $a^{-2}(2^{-b}/a)^2 = a^{-4}2^{-2b}$ is an upper bound on the number of ways to choose $\hat{x}_1$ and $\hat{x}_2$ from Ω_a. Hence, we get the following upper bound:

$$O(1) k^2 2^{-7b/4} C_{k-2} a^{-\frac{15k}{8}}\,.$$

It remains to sum over the possible values of b, i.e., $\log_2(\epsilon^{-1}) \leq b \leq \log_2(a^{-1})$, which gives the desired result.

Modulo boundary issues that are easily dealt with, this concludes the proof of the lemma, which in turn implies the proposition. □

4.4 Consequences of this approach

This proof of Theorem 1.2 through the study of the moments of m^a sheds some light on Φ^∞. For example it enables in some cases to explicitly compute the variance of m_∞. Indeed, in the full plane $\mathbb{C}$, if one looks at $\langle \Phi, 1_A \rangle$, then from the work of [CHI12] or [D11], we get that

$$\mathbb{E}\big[\langle \Phi_{\mathbb{C}}, 1_A \rangle^2\big] \quad = \quad C \iint_A \frac{1}{|x-y|^{1/4}} dxdy\,, \tag{4.6}$$

Here C is a constant which can be computed explicitly thanks to the formula (see Theorem 1.6) by T.T. Wu ([MW73]). Therefore, the second moment of $\langle \Phi_{\mathbb{C}}, 1_A \rangle$ can be computed numerically or exactly depending on the set A.

5 Conformal covariance structure

The scaling limit Φ^∞ of the magnetization field that we identified in the previous two sections has the following conformal covariance property.

Theorem 5.1 (Conformal covariance of Φ^∞)**.** *Let $\Omega, \tilde{\Omega}$ be two simply connected domains of the plane (not equal to $\mathbb{C}$) and let $\phi : \Omega \to \tilde{\Omega}$ be a conformal map. Let $\psi = \phi^{-1}$ be the inverse conformal map from $\tilde{\Omega} \to \Omega$. Let Φ^∞ and $\tilde{\Phi}^\infty$ be the continuum magnetization fields respectively in Ω, $\tilde{\Omega}$. Then, the pushforward distribution $\phi * \Phi^\infty$ of the random distribution Φ^∞ has the same law as the random distribution $|\psi'|^{15/8}\tilde{\Phi}^\infty$, where the latter distribution is defined as*

$$\langle |\psi'|^{15/8}\tilde{\Phi}^\infty, \tilde{f}\rangle := \langle \tilde{\Phi}^\infty, w \mapsto |\psi'|^{15/8}(w)\tilde{f}(w)\rangle \,,$$

for any test function $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$.

Let us illustrate this conformal invariance property in the particular case of the renormalized magnetization in squares of various scales.

Corollary 5.2. *Let m^∞ be the scaling limit of the renormalized magnetization in the square (i.e., $m^\infty = \langle \Phi^\infty, 1_{[0,1]^2}\rangle$). For any $\lambda > 0$, let m^∞_λ be the scaling limit of the renormalized magnetization in the square $[0, \lambda]^2$. Then one has the following identity in law:*

$$m^\infty_\lambda \overset{d}{=} \lambda^{15/8}\, m^\infty \,. \tag{5.1}$$

Let us discuss how to prove Theorem 5.1.

1. If one wants to follow the setup of our first proof (Section 3), then the conformal covariance property, Theorem 5.1, is proved exactly in the same fashion as Theorem 6.1 in [GPS10] on the conformal covariance of the *pivotal measures* for critical percolation on the triangle lattice, except here one would have conformal covariance of the ensemble of FK area measures.

2. If one wants to follow the setup of our second proof (Section 4), then Theorem 5.1 is even easier to obtain, since it follows easily from the conformal covariance properties of the k-point functions established in the main result, Theorem 1.3, of [CHI12].

6 Without assuming T. T. Wu's result

The purpose of this section is to briefly explain how to adapt our proofs if one does not want to rely on T. T. Wu's result, Theorem 1.6. In this case, as explained in Subsection 1.2, one would need to renormalize our fields by

$$\Theta_a := a^2\, \varrho\,(a)^{-1/2} \;, \tag{6.1}$$

instead of $\Theta_a = a^{15/8}$.

6.1 Adapting the first proof (Section 3)

Let us point out here that it is not a priori needed to have an exact rescaling of the form $a^{15/8}$ if one wants to obtain our main result, Theorem 1.2. For example, this situation arises in [GPS10], where the four-arm event is only known up to possible logarithmic corrections. Therefore, in order to build the *pivotal measures* there, it is not possible to assume a renormalization of the discrete counting measure by $\eta^{3/4}$; instead, a more cumbersome renormalization of $\eta^2\alpha_4(\eta,1)^{-1}$ is needed—see [GPS10] for more details. In the present work, the same technology as in [GPS10] would enable us to prove Theorem 1.2 without relying on T. T. Wu's result.

Yet, some of the present proofs would need to be slightly modified and some quantitative lemmas (such as Lemma 2.6 for example) would need to be changed. Let us point out that we would have at our disposal the following useful bound on the one-arm event:

$$C\, n^{-1/2} \le \alpha_1^{\mathrm{FK}}(n) \le n^{-\alpha}\,, \tag{6.2}$$

for some exponent $\alpha > 0$. The lower bound follows from Smirnov's observable (see [DHN11]) while the upper bound follows from the RSW theorem in [DHN11]. Such bounds are enough to carry the proof from [GPS10] through (except for the conformal covariance property, Theorem 5.1, which needs at least an $SLE_{16/3,16/3-6}$ computation for the one-arm event).

6.2 Adapting the second proof (Section 4)

The second proof is easier to adapt, since the results in [CHI12] are stated precisely with the renormalization factor $\Theta_a = a^2\,\varrho(a)^{-1/2}$. The proof of Proposition 4.5 works as before. Of course, Proposition 4.9 would be stated in a less quantitative manner, but using for example the above estimate (6.2), one could still handle the local k-tuples, which would give us the desired result.

7 Properties of the limiting magnetization field Φ^∞

In this last section, we wish to list some interesting properties satisfied by the magnetization field Φ^∞ which will be proved in [CGN] as well as some results on the near-critical behavior of the Ising model along the h-direction which appear or will appear in [CGN12, CGN].

1. The random field Φ^∞ is not Gaussian; if m_Ω^∞ denotes the scaling limit of the renormalized magnetization in a bounded domain Ω, then there exists a constant $c = c_\Omega > 0$ such that

   $$\log \mathbb{P}\big[m_\Omega^\infty > x\big] \underset{x\to\infty}{\sim} -c\, x^{16}\,.$$

   Furthermore, it will be shown that the constant $c = c_\Omega$ depends on the domain but does not depend on the boundary conditions.

2. The probability density function of m_Ω^∞ is smooth as a consequence of the following quantitative bound on its Fourier transform: $\forall t \in \mathbb{R}$,

   $$|\mathbb{E}_\Omega\big[e^{i\,t\,m^\infty}\big]| \le e^{-C|t|^{16/15}}\,, \tag{7.1}$$

   for some constant $C > 0$.

3. In [CGN], it will be shown that the Ising model on the rescaled lattice $a\mathbb{Z}^2$ with renormalized external magnetic field $h_a := h\, a^{15/8}$ has a *near-critical* (or *off-critical*) scaling limit as $a \searrow 0$. This near-critical limit is no longer scale-invariant but is conformally covariant instead and has exponential decay of its correlations.

4. Finally, in [CGN12] we prove that the average magnetization $\langle \sigma_0 \rangle$ of the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ and with external magnetic field $h > 0$ is such that

$$\langle \sigma_0 \rangle_{\beta_c, h} \asymp h^{\frac{1}{15}} \,. \tag{7.2}$$

A Appendix: first and second moments for the magnetization

The main purpose of this appendix is to prove the following proposition on the first and second moments of the magnetization in a bounded smooth domain Ω. (In fact, to simplify the notation, we will only prove it in the case where Ω is a square domain; see Proposition A.2). Along the way, we will also prove some useful bounds on the one-arm event in critical FK percolation (Lemma A.3). Let us point out that in this appendix, we do not need to assume Theorem 1.6.

Proposition A.1. *Let Ω be a bounded smooth domain of the plane. Let $M^a_\Omega = M^a$ be the (non-renormalized) magnetization*

$$M^a = \sum_{x \in \Omega_a} \sigma_x \,.$$

There is a constant $C > 0$ such that for each mesh size $a > 0$, one has

(i) $\mathbb{E}^+\big[M_a\big] \le C\, a^{-2} \sqrt{\varrho(a)}$

and

(ii) $\mathbb{E}^+\big[(M_a)^2\big] \le C a^{-4}\, \varrho(a)$.

(Obviously here (i) follows from (ii) using Cauchy-Schwarz). For simplicity of presentation, we will prove this result only in the particular case where Ω is a square domain. Furthermore, in order to simplify the notation in the proof, we will work with a non-renormalized lattice. Before restating the above proposition in this setting, let us introduce the following notation: for any $N \ge 1$, let

$$\rho(N) := \mathbb{E}_{\mathbb{Z}^2}\big[\sigma_{(0,0)} \sigma_{(N,N)}\big] \,. \tag{A.1}$$

As such, $\rho(N)$ is related to $\varrho(a = \sqrt{2}(N)^{-1})$, where $\varrho(a)$ was defined in (1.1).

We will show the following proposition.

Proposition A.2. *For any $N \ge 1$, let Λ_N be the square $[-N, N]^2$ and let M_N be the magnetization in Λ_N, i.e.,*

$$M_N := \sum_{x \in \Lambda_N} \sigma_x \,.$$

Then, there is a constant $C > 0$ such that for all $N \ge 1$,

(i) $\mathbb{E}^+\big[M_N\big] \le C\, N^2\, \rho(N)^{1/2}$

and

(ii) $\mathbb{E}^+\big[M_N^2\big] \le C\, N^4\, \rho(N)\,.$

The proof of the proposition relies on the following lemma, which already appeared in [CN09]. To be self-contained, we include a proof here. (Also, the lemma below includes more than what is actually needed for Proposition A.2 but it will be useful for future reference.) We denote by $\mathbb{P}_{p_c}^{\mathrm{free}}\big[\cdot\big]$ (resp, $\mathbb{P}_{p_c}^{+}\big[\cdot\big]$) the critical FK percolation measure with free (resp., wired) boundary conditions.

Lemma A.3. *There exists a constant $C < \infty$ such that*

$$\begin{cases} \frac{1}{C}\sqrt{\rho(N)} \le \mathbb{P}_{p_c}^{+}\big[0 \leftrightarrow \partial\Lambda_N\big] \le C\sqrt{\rho(N)} \\ \frac{1}{C}\sqrt{\rho(N)} \le \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N\big] \le C\sqrt{\rho(N)} \\ \rho(N) \le C\,\rho(2N) \end{cases}$$

Proof. To derive the first two parts of the lemma, it is clearly enough to prove the following inequality for some constant $C < \infty$:

$$\frac{1}{C}\sqrt{\rho(N)} \le \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N\big] \le \mathbb{P}_{p_c}^{+}\big[0 \leftrightarrow \partial\Lambda_N\big] \le C\sqrt{\rho(N)}\,.$$

Let us first handle the LHS: clearly, using FKG, one has

$$\rho(N) \le \mathbb{P}_{p_c}^{+}\big[0 \leftrightarrow \partial\Lambda_{N/2}\big]^2\,.$$

Now we wish to show that

$$\mathbb{P}_{p_c}^{+}\big[0 \leftrightarrow \partial\Lambda_{N/2}\big] \le c\mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N\big]\,, \tag{A.2}$$

for some constant $c < \infty$. This can be seen as follows: let R_N be the event that there is open circuit in the annulus $\Lambda_{N/2} \setminus \Lambda_{N/4}$, then

$$\begin{aligned} \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N\big] &\ge \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N; R_N\big] \\ &\ge \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_N \bigm| R_N\big] \\ &\ge \mathbb{P}_{p_c}^{+}\big[0 \leftrightarrow \partial\Lambda_{N/2}\big]\mathbb{P}_{p_c}^{+}\big[\partial\Lambda_{N/4} \leftrightarrow \partial\Lambda_N\big]\,, \end{aligned}$$

which concludes the proof of (A.2) by using RSW from [DHN11]. Altogether this proves the LHS inequalities in the first two parts of Lemma A.3. The RHS is proved along the same lines. Namely, one clearly has by FKG that

$$\rho(N) \ge \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_{N/2}\big]^2\,.$$

Now obviously, $\mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_{N/2}\big] \ge \mathbb{P}_{p_c}^{\mathrm{free}}\big[0 \leftrightarrow \partial\Lambda_{2N}\big]$ and thus, using again (A.2), this concludes the proof of the first two parts of Lemma A.3. It is easy to see from the above computation that, possibly by changing the value of C, one can get the last part of Lemma A.3. $\square$

Proof of Proposition A.2. Even though, as pointed out above, property (i) follows from property (ii) by Cauchy-Schwarz, we will give a detailed proof of (i) and only briefly highlight how to deal with (ii).

We divide the domain Λ_N into $n \asymp \log_2 N$ disjoint annuli $A_0, \ldots, A_n$ such that for each $i \in [0, n]$, the vertices in A_i are at distance 2^i (up to a factor of 2) from the boundary $\partial\Lambda_N$. This decomposition gives us

$$\begin{aligned}\mathbb{E}^+\big[M_N\big] &= \sum_{0\le i\le n}\sum_{x\in A_i}\mathbb{P}^+\big[x \leftrightarrow \partial\Lambda_N\big] \\ &\le O(1)\sum_{0\le i\le n}\#\{A_i\}\,\mathbb{P}^+\big[0\leftrightarrow \partial\Lambda_{2^i}\big] \\ &\le O(1)\sum_{0\le i\le n} N\,2^i\,\mathbb{P}^+\big[0\leftrightarrow\partial\Lambda_{2^i}\big]\,.\end{aligned}$$

Now, one has that for any $i \le n$,

$$\begin{aligned}\mathbb{P}^+\big[0\leftrightarrow\partial\Lambda_N\big] &\ge \mathbb{P}^{\text{free}}\big[0\leftrightarrow\partial\Lambda_{2^i}\big]\mathbb{P}^{\text{free}}\big[\partial\Lambda_{2^i}\leftrightarrow\partial\Lambda_N\big] \\ &\ge 1/C^2\mathbb{P}^+\big[0\leftrightarrow\partial\Lambda_{2^i}\big]\mathbb{P}^{\text{free}}\big[\partial\Lambda_{2^i}\leftrightarrow\partial\Lambda_N\big]\,,\end{aligned}$$

from Lemma A.3. Continuing the above computation, one obtains

$$\mathbb{E}^+\big[M_N\big] \le O(N)\sum_{0\le i\le n} C^3\,2^i\,\frac{\sqrt{\rho(N)}}{\mathbb{P}^{\text{free}}\big[\partial\Lambda_{2^i}\leftrightarrow\partial\Lambda_N\big]}\,.$$

It is known from [DHN11], Proposition 24, that $\mathbb{P}^{\text{free}}\big[\partial\Lambda_{2^i}\leftrightarrow\partial\Lambda_N\big] \ge c(2^i/N)^{1/2}$ for some constant $c > 0$. This gives

$$\begin{aligned}\mathbb{E}^+\big[M_N\big] &\le O(1)N\sum_{0\le i\le n} 2^{i/2}N^{1/2}\sqrt{\rho(N)} \\ &\le O(1)N^2\sqrt{\rho(N)}\,,\end{aligned}$$

which completes the proof of condition (i).

The proof for the second moment (ii) follows exactly the same lines except that the combinatorics is slightly more tedious. As an indication, let us give two upper bounds which are useful to carry out the computation properly: if $x, y \in \Lambda_N$ are such that $l := |x-y| \le \min(d(x,\partial\Lambda_N), d(y,\partial\Lambda_N))$, then one has

$$\mathbb{E}^+\big[\sigma_x\sigma_y\big] \le O(1)\mathbb{P}^+\big[0\leftrightarrow\partial\Lambda_l\big]^2\mathbb{P}^+\big[\partial(z+\Lambda_{2l})\leftrightarrow\partial\Lambda_N\big]\,, \tag{A.3}$$

where z is the midpoint between x and y. If, on the other hand, one of the points is close to the boundary, in the sense that $|x-y| > \min(d(x,\partial\Lambda_N), d(y,\partial\Lambda_N))$, then one can dominate $\mathbb{E}^+\big[\sigma_x\sigma_y\big]$ by $O(1)\mathbb{P}^+\big[x\leftrightarrow\partial\Lambda_N\big]\mathbb{P}^+\big[y\leftrightarrow\partial\Lambda_N\big]$. □

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Federico Camia
Department of Mathematics, Vrije Universiteit Amsterdam and NYU Abu Dhabi
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Christophe Garban
ENS Lyon, CNRS
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Charles M. Newman
Courant institute of Mathematical Sciences, New York University, New York, NY 10012, USA
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